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TRANSIENT SOUND PROPAGATION IN A SIMPLE MODEL OF A TRIPLE-LAYERED MEDIUM

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PREFACE

This report discusses a simple model of long-range sound transmission. It is part of RAND's continuing study of the atmospheric waves generated by nuclear explosions. A later report is planned which will apply the theory developed in this report to the explanation of some of the characteristic features of the waveforms detected at large distances.

This report should be of interest to agencies and contractors concerned with the detection of nuclear explosions. It should also be of interest to other research workers who are studying the phenomena associated with wave propagation in inhomogeneous media.

SUMMARY

A theoretical study has been made of the transient sound propagation from a point source in a triple-layered medium consisting of a homogeneous fluid layer sandwiched between two similar half spaces of the same density but of higher sound speed. The source spectrum is such that a large number of normal modes are excited. Both source and receiver are assumed to be in the same half space. The general solution consists of a direct wave, a wave reflected from the near edge of the center layer, a lateral wave, and the normal and complex mode waves. At large distances r the normal mode waves dominate. It is found that one may relate frequency, phase velocity, group velocity, and excitation amplitudes parametrically, and, on the basis of this, the variation of the characteristics of the normal modes with mode number n is studied. Similar methods can be used to study the complex modes. Each normal mode is found to carry a limited band of frequencies--the band width for higher order modes being approximately proportional to n^{-1}. Unless r is extremely large, the amplitudes of the Airy phases for the higher order modes will be negligible. On the basis of the study, one may give a qualitative description of the waveforms. One effect of placing source and receiver outside the center layer is to accentuate lower order modes and the lower frequency portion of the source spectrum.

CONTENTS

I. INTRODUCTION

In the hierarchy of guided wave problems in stratified media, that of giving a theoretical interpretation to the transient wave-forms observed at large distances from a high frequency source is generally considered to be among the more difficult. Although both of the standard mathematical approaches generally utilized in guided wave problems (namely the so-called normal mode[1-4] and ray theory approaches[5-8]) may in principle be applied, neither seems to lead to an easily interpretable solution. The reasons for this difficulty are readily understandable. Since the distance from the source is large, the number of individual rays which must be summed will also be large. Also, since the frequency band of the source lies above the cutoff frequencies of a large number of normal modes, one must generally superimpose many modes.

Intuitively, one would feel that the normal mode approach is the better of the two, as it is an asymptotic theory which enables one to separate out at the very beginning that portion of the wave which will predominate at large distances. However, for even relatively simple models of stratified media, the calculation of the pertinent characteristics (i.e., group velocities and excitation amplitudes) of the normal modes must be done numerically. (The only exception of which we are aware is the model in which the sound speed varies as in Epstein's symmetrical layer.[9,10]) As long as one is justified in considering one or a small number of modes, this does not present insurmountable difficulties. Indeed, some very convenient[11,12] methods have been devised for accomplishing

this for models of various sophistication. Nevertheless, within this context, the study of the systematic variation of mode characteristics with mode number requires the numerical calculation of the characteristics for a large number of modes.

To minimize such difficulties, it is convenient to begin with the study of a very simple model and to consider this model in some detail. The purpose of this paper is to report on the results of such a study.

The model we chose was that of a homogeneous fluid layer sandwiched between two similar homogeneous half spaces of higher sound velocity. Mathematically, this model is very similar to that originally proposed by Pekeris[1] for the study of sound propagation in shallow water. For example, the dispersion characteristics of the second, fourth, etc., modes for our model are identical to those of the first, second, etc., modes for Pekeris' model. Our principal reason for considering this particular model rather than Pekeris' is that our interest is primarily in sound propagation in the atmosphere rather than in the ocean, and we felt that our model would hold a greater relevancy to this subject.

Whether or not this model is relevant to the understanding of sound propagation in the atmosphere is questionable at the present. Our belief in its relevancy is based on the fact that the first temperature minimum in the atmosphere[13] lies some distance ($\sim$ 20 km) above the ground, and that the temperature maximum at the 50 km inversion is less than that at the ground. It therefore seems reasonable to believe that higher frequency waves trapped in the lower atmospheric sound channel will be little affected by the

presence of the ground. Naturally, this will not be true for the very low frequency waves (the so-called gravity waves), which have recently been studied by a number of workers,[14-17] but should become of increasing validity when one considers higher frequencies. A model such as ours, though vastly oversimplified, might be helpful in acquiring a qualitative understanding of some aspects of the physical phenomena. In the present paper, however, we shall not explore the possibility of the relevancy of our model to this subject.

To simplify our model as much as possible, we have assumed that the middle layer and surrounding half spaces all have the same density, though we have otherwise (to as large an extent as practicable) avoided any numerical specification of the ratio of the two sound speeds or the width of the center layer. We have tried to treat the problem in some generality and to make maximum use of analytic techniques rather than to rely solely on numerical calculations. For the most part, we have limited our consideration to the case where both source and receiver are outside the channel and on the same side of the channel (i.e., where both lie in the same half space). The mathematical formulas obtained for other source and receiver positionings are appreciably different, and it was felt that a consideration of all possible cases would unduly lengthen the paper.

This specification was also motivated by the desire that our model have some relevancy to the study of propagation in the atmosphere--where both source and receiver are commonly near the

ground and hence (in a certain sense) outside the channel. This departs from the usual procedure of placing both inside the channel, as is done in the majority of papers on wave propagation in the ocean. The configuration we consider has received little attention, although Officer does briefly discuss this topic in his book.[18]

In our analysis of the problem, we have been aided by the fact that the characteristics of the normal modes may all be related parametrically to the frequency by the introduction of a variable μ. This is possible even if one takes the density of the center layer to be different from that of the surrounding half spaces, although considerable simplification is achieved by taking all densities equal. Studying the parametric equations enables us to derive several properties concerning the mode characteristics and furnishes us with a convenient method of studying the variation of these characteristics with mode number.

Placing the source and receiver outside the channel turns out to have some interesting consequences. The excitation amplitudes are of appreciable magnitude over only a limited range of frequencies for each mode, and the overlap of the frequency bands for different modes is small. For each normal mode the amplitudes of the "ground wave" are generally much larger than those of the "water wave" or the Airy phase unless the distance from the source to receiver is exceptionally large. The maximum amplitude of each mode decreases with increasing mode number, giving the impression that the model favors lower frequencies. These effects shall be discussed in some detail.

II. THE FORMAL SOLUTION

The model we shall consider is conveniently described by specifying the dependence of the speed of sound c on the vertical coordinate $\underline{z}$. For $z > H/2$ or $z < -H/2$ (i.e., in the two half spaces) the sound speed is c_o. In the center layer, where $-H/2 < z < H/2$, the sound speed is c_1, where $c_1 < c_o$. The density ρ is constant throughout the media. A point source of sound is located on the z-axis at a distance z_o from the plane of symmetry (Fig. 1). In the immediate vicinity of the source the velocity potential behaves like an outgoing spherical wave,

$$\phi = f(t-R/c)\ /R, \tag{2.1}$$

where R is the distance from the source and f(t) is the time dependence of the source. In general, ϕ is the solution of the simple wave equation and is continuous with continuous gradients everywhere except at the source.

The techniques of finding a formal solution to problems of this type have been discussed at some length in the literature.(1-4) The general procedure is to express it as a double integral over frequency ω and wave number k. In treating the case when both source and receiver are in the same half space, however, it is convenient first to extract a direct wave and a wave reflected from the near edge of the guide, such that

$$\psi = f(t-(R_D/c_o))/R_D$$

$$- f(t-(R_I/c_o))/R_I$$

$$+ \psi' \qquad , \qquad (2.2)$$

where

$$R_D = [r^2 + (z-z_o)^2]^{1/2}$$

and

$$R_I = [r^2 + (|z+z_o| - H)^2]^{1/2}.$$

The extraction conforms to our intuitive notion that these waves should be present and aids in shortening our presentation. A tentative justification for assigning a negative sign to the second term can be given if one notes that the Rayleigh reflection coefficient[19] for angles near grazing incidence is nearly -1. Since we are interested in the case of large r, any wave reflected from the near edge of the guide must be considered as being near grazing incidence.

After the extraction of the two waves, the remaining term ψ' may be written

$$\psi' = 2\mathrm{Re} \int_0^\infty g(\omega)e^{-i\omega t}d\omega \int_{-\infty}^\infty H_0^{(1)} (kr)Ukdk, \qquad (2.3)$$

where $g(\omega)$ is the Fourier transform of $f(t)$, and U is independent of the radial coordinate r. For our assumed receiver-source configuration, U is essentially a function of three quantities: β_o, β_1, and D, where

$$\beta_o^2 = (\omega/c_o)^2 - k^2, \tag{2.4a}$$

$$\beta_1^2 = (\omega/c_1)^2 - k^2, \tag{2.4b}$$

and

$$D = \frac{2}{H}\,(|z+z_o| - H). \tag{2.5}$$

The phase of β_o (or β_1) is taken to be 0 or $\frac{\pi}{2}$, depending on whether k^2 is less or greater than $(\omega/c_o)^2$ (or $(\omega/c_1)^2$). The quantity D represents the sum of the distances of the source and receiver from the near edge of the guide in units of 1/2 H. Explicitly,

$$U = \frac{i}{(\beta_o+\beta_1)}\,\frac{(1-VQ^2)}{(1-V^2Q^2)}\quad e^{i\beta_o DH/2}\,, \tag{2.6}$$

where

$$V = (\beta_o - \beta_1)/(\beta_o + \beta_1) \tag{2.7}$$

and

$$Q = e^{i\beta_1 H} \tag{2.8}$$

The derivation of (2.6) proceeds along standard lines[1-4] and is omitted in the interests of brevity.

The integration over k in Eq. (2.3) is properly a contour integration in the complex plane. To insure causality, the path must pass above all poles lying on the negative real axis and below all poles lying on the positive real axis. Similarly, the integration over ω in (2.3) is understood to pass above all poles on the real axis.

Since the normal mode method rests on a contour deformation in the k plane, it is necessary to extend the definition of the

integrand to include complex values of k. This is accomplished by placing branch lines and requiring that the integrand be analytic everywhere except at these branch lines and at its poles. The placing of the branch lines is standard. One branch line, made necessary by the definition of the Hankel function, extends vertically downwards from the origin. Those necessary for the unique specification of the phase of β_o extend vertically upwards from the branch point at $k = \omega/c_o$ and vertically downwards from $k = -\omega/c_o$. Branch lines for β_1 are not essential, as U is invariant if β_1 is replaced by $-\beta_1$. However, since U is not explicitly given as a function of β_1^2, it is convenient to add branch lines extending upwards from ω/c_1 and downwards from $-\omega/c_1$ to avoid any inconsistencies. It is readily seen that our previous specification of the phases of β_o and β_1 along the real axis is consistent with the placing of our branch lines.

With the contour deformation appropriate to the method of normal modes, the path in the k plane is transformed into a sum of contour integrals around the poles lying in the upper half plane and on the positive real axis plus a branch line integral around the line extending upwards from ω/c_o. This contour is sketched in Fig. 2.

With this deformation, the quantity ϕ may be decomposed into three terms,

$$\phi' = \phi_{NM} + \phi_{CM} + \phi_L , \tag{2.9}$$

where the first term (normal mode solution) represents the contribution from the poles on the real axis, the second term (complex mode

solution) represents the contribution from the poles in the upper half plane, and the third term (lateral wave) represents the contribution from the branch line integral.

The contour integrals around the poles may be formally evaluated by the method of residues. A simplified expression for the residues may be derived if one notes that the poles coincide with the roots of the equation

$$1 - V^2 Q^2 = 0. \tag{2.10}$$

Then, one may show that, near a pole, U is of the form

$$U = \frac{G}{kH(k-k_p)}, \tag{2.11}$$

where k_p denotes the location of the pole and

$$G = \left\{ \frac{-i\beta_o H/2}{1 - i\beta_o H/2} \right\} \left\{ \frac{\beta_1^2}{\beta_1^2 - \beta_o^2} \right\} e^{i\beta_o DH/2} \tag{2.12}$$

is to be evaluated at $k = k_p$. The result of the integration around a pole may readily be obtained by taking $2\pi i$ times the limit of the product of $(k-k_p)$ and the integrand as k approaches k_p.

Since the positions of the poles will generally change continuously with ω, each pole may be assigned an index n (for those on the real axis) or m (for those above the real axis), where n, m = 1,2,3, etc. The manner of assigning these indices, or mode numbers, may be made unambiguous and will be described in the next section. As is well-known, poles corresponding to a given mode

number will not necessarily be present for all values of ω. In general they will be present over some range of frequencies, and one may speak of a low frequency or high frequency cutoff (ω_c or ω_c') for each mode. The poles on the real axis (i.e., the normal modes) will have no high frequency cutoff, but those off the real axis (i.e., the complex modes) will in general have <u>both</u> low frequency and high frequency cutoffs. (The latter shall be shown to be true in the next section and in Appendix A.)

Up to now, the method we have indicated for evaluating the normal mode or complex mode solutions is first to perform a sum over poles and then integrate over frequency. With the introduction of the concept of mode number it is possible to interchange this sequence. Thus the formal expression for ψ_{NM} may be given in the form

$$\psi_{NM} = \frac{4\pi}{H} \operatorname{Re} \sum_{n=1}^{\infty} \int_{\omega_c}^{\infty} iGH_0^{(1)}(kr)g(\omega)e^{-i\omega t}d\omega, \tag{2.13}$$

where k is $k_n(\omega)$ and the cutoff frequency ω_c depends on n. A similar expression holds for ψ_{CM}--the chief formal distinction being that the integration for each mode has a high frequency cutoff ω_c'.

The remaining portion of the solution, the lateral wave, is expressible in the form

$$\psi_L = 2\operatorname{Re} \int_0^{\infty} g(\omega)e^{-i\omega t}d\omega \int_{\omega/c_0}^{\omega/c_0+i\infty} [U(\beta_0)-U(-\beta_0)]H_0^{(1)}(kr)kdk,$$

where we have changed the integration over k from the original path (which proceeded down one side of the branch line and up the other) to a path from the branch point upwards along the right side of the branch line. This was accomplished by making use of the fact that the only formal distinction in the integrand on the two sides of the branch line was a difference of π in the phase of β_0. The value of β_0 to be utilized in (2.14) is that on the right side of the branch line.

The decomposition of the velocity potential into a direct wave, reflected wave, normal mode solution, complex mode solution, and lateral wave as outlined above is (subject to some mathematical fine-points) a rigorous expansion of the solution along the lines of the normal mode approach.

III. LOCATION OF THE POLES IN THE COMPLEX PLANE

One barrier in the utilization of the formal solution given in the previous section is that of finding the expressions $k_n(\omega)$, i.e., of locating the positions of the poles. These are the roots of the transcendental equation (2.10), for which, in general, explicit solutions in terms of elementary functions cannot be obtained. We shall here introduce a method which will facilitate the study of these roots and which is convenient for numerical computations. We shall not restrict our attention to the real roots alone (although these are of primary interest), as our method makes a study of the complex roots feasible without an undue amount of effort.

Let us first note that the difference $\beta_1^2 - \beta_o^2$ is independent of the wave number k. In particular, we have

$$\beta_1^2 - B_o^2 = (2/H)^2(\omega/\bar{\omega})^2 = (\omega/c_o)^2 A^2, \tag{3.1}$$

where

$$\bar{\omega} = (2/H)c_o c_1(c_o^2 - c_1^2)^{-1/2} \tag{3.2}$$

is a frequency and

$$A^2 = (c_o/c_1)^2 - 1 \tag{3.3}$$

is a dimensionless constant which characterize the model. (Both A and $\bar{\omega}$ are real since we are assuming $c_o > c_1$.)

The lack of dependence of (3.1) on k suggests the following substitution:

$$\beta_1 = (2B/H)\cos\mu \tag{3.4a}$$

$$\beta_0 = i(2B/H)\sin\mu\ , \tag{3.4b}$$

where we have abbreviated $B = \omega/\bar{\omega}$. In terms of the variable μ just introduced, the equation (2.10) would appear as

$$\exp\left[4i(B\cos\mu - \mu)\right] = 1, \tag{3.5}$$

which has an appealing simplicity when contrasted with its original form in terms of k.

There are certain mathematical pitfalls involved in the utilization of (3.5) instead of (2.10), and some care should be exercised. We shall first restrict the real part of μ to lie between $-\pi$ and π. With this restriction and the definitions (3.4), a unique value of μ is associated with each point in the k-plane. We shall consider this association in some detail.

The k-plane may be imagined as consisting of eight regions--the boundaries of these regions being formed by the real and imaginary axes and the branch lines extending from $\pm\ \omega/c_0$ and $\pm\ \omega/c_1$. Although it is laborious to make a detailed specification of β_1 and β_0 for every point, it is a relatively simple matter to specify the quadrants in which their phases lie for each of the eight regions. For example, in the region to the right of the branch line extending from ω/c_1 and above the real axis, the phases of both β_0 and β_1 will lie in the second quadrant. Providing this specification

is a routine problem in the theory of complex variables. Furthermore, if we know the quadrants in which these phases lie, we may determine via equations (3.4) the sign of the imaginary part of μ and the quadrant in which its real part lies. In the region mentioned above, for example, μ_I is negative and μ_R is in the second quadrant. We have carried out this analysis for each of the eight regions. The results are given in Fig. 3.

Let us now note that equation (3.5) may be satisfied only if the imaginary part of $B \cos \mu - \mu$ is zero. When one writes out the imaginary part explicitly, he discovers this can be true only if either $\mu_I = 0$ or $\sin \mu_R$ is negative. Referring to Fig. 3, we see that, if a root of (3.5) is to correspond to a value of k which satisfies (2.10), this value of k must lie either on the real axis between ω/c_o and ω/c_1 or between $-\omega/c_o$ and $-\omega/c_1$, or else it must lie in one of the two regions which are shaded in Fig. 3. Thus when studying the real roots of (2.10), we shall restrict our attention to roots of (3.5) for which $\mu_I = 0$ and μ_R is between 0 and $\pi/2$. When studying the complex roots we restrict our attention to roots of (3.5) for which $\mu_I < 0$ and μ_R is between $-\pi/2$ and 0.

Since (2.10) is invariant under replacing k by -k, the roots are symmetrically located in the k-plane. We shall accordingly limit our consideration to those roots in the first quadrant and on the positive real axis. By utilization of either (3.4a) or (3.4b) and the definitions of β_1^2 and β_o^2, one may formally express k^2 in terms of μ:

$$k^2 = (\omega/c_o)^2(1 + A^2 \sin^2\mu). \tag{3.6}$$

If μ lies in either of the two regions defined in the preceding paragraph, the above equation will lead to a unique value of k in the first quadrant or on the positive real axis. If μ is real and between 0 and $\pi/2$, the value of k computed in this manner will always lie between ω/c_o and ω/c_o, and any such root of (3.5) will correspond to a root of (2.10). On the other hand, if $\mu_I < 0$ and $-\pi/2 < \mu_R < 0$, the k computed from (3.6) will not necessarily lie to the left of the branch line which extends from ω/c_o. If it lies to the right of the branch line, the equations (3.4) will be incompatible with the definitions of β_1 and β_o. This possibility can occur, since every μ will not necessarily correspond to a point in the k-plane.

The correct procedure for utilizing (3.5) is readily apparent. To find all of the roots of (2.10) lying on the positive real axis one first finds all the roots of (3.5) for which μ is real and between 0 and $\pi/2$, and then computes k from (3.6). To find all of the complex roots lying in the first quadrant one first finds all roots of (3.5) for which $\mu_I < 0$ and $-\pi/2 < \mu_R < 0$ and computes a k lying inthe first quadrant via (3.6) for each such μ. He then eliminates all complex k's which do not lie to the left of the branch line. It is easily verified that the roots obtained in this manner and their negatives (inversions through the origin) constitute all of the roots of (2.10).

It should be mentioned at this point that the parameter μ has no immediately apparent physical significance. It is merely a parameter which aids in studying the solution of the problem.

The Real Roots

Equation (3.5) is satisified only if $B \cos \mu - \mu$ is a multiple of $\pi/2$. Thus μ must satisfy an equation of the form

$$B \cos \mu - \mu = (1/2)(n-1)\pi, \tag{3.7}$$

where n is an integer. We shall consider μ to be real and shall tentatively identify n as the mode number for the normal modes.

There will be solutions of (3.7) between 0 and $\pi/2$ only if $n \geq 1$ and $B \geq 1/2\ (n-1)$. The low frequency cutoff for the n-th normal mode is therefore

$$\omega_c = \frac{1}{2}(n-1)\pi\bar{\omega}. \tag{3.8}$$

If n is an even integer, this will be the same as that for the n/2-th mode in Pekeris' model.

If ω lies between the cutoff frequencies of the $\bar{n}$-th and $(\bar{n}+1)$-th modes, there will be one and only one solution of equation (3.7) for each value of n between 1 and $\bar{n}$ inclusive, and there will be $\bar{n}$ roots k_n of equation (2.10) which lie on the real axis between ω/c_o and ω/c_1.

As might be anticipated, we cannot solve (3.7) explicitly for μ and cannot obtain an explicit formula for k_n. Nevertheless, we can solve for B (or ω), and it is there that its utility lies. The two equations,

$$B = \omega/\bar{\omega} = [\tfrac{1}{2}(n-1)\pi + \mu]/\cos \mu \tag{3.9}$$

and

$$k = (\omega/c_o)(1 + A^2\sin^2\mu)^{1/2}, \tag{3.10}$$

which are obtainable from (3.7) and (3.6), respectively, relate k and ω for a given n parametrically through μ. Thus the line $k_n(\omega)$ in the k-ω plane may be obtained in principle by allowing μ to run from 0 to $\pi/2$ and computing ω via (3.9) and then k via (3.10) for each value of μ. The equations (3.9) and (3.10) therefore afford a complete description of the relationship between k and ω for the n-th normal mode. Furthermore, as shall be shown in Section V, they furnish a convenient starting point for analytic study of the dispersion characteristics of the normal modes.

It is to be noted that we could have obtained similar equations even if we had not assumed the density of the center layer to be the same as that of the two half spaces. With the same defintion of μ as given previously, k would also be given by (3.10). Instead of (3.9), however, we would find

$$B = \frac{(n-1)\pi/2 + \tan^{-1}[(\rho_o/\rho_1)\tan\mu]}{\cos\mu},$$

where ρ_o is the density in the half spaces and ρ_1 is the density in the center layer. The relative simplicity achieved by equating ρ_o and ρ_1 is clear.

The Complex Roots

The study of the roots off the real axis may be greatly facilitated by use of the auxiliary equation (3.5), although the

resulting mathematical expressions are much more complicated than in the case of the real roots. Here we summarize some of the results of such a study. The mathematical methods are discussed in Appendix A.

The complex roots in the first quadrant lie along a line. The path of this line may be described parametrically by expressing both the real and imaginary parts of $k = k_R + i\,k_I$ in terms of the imaginary part of μ. This line of roots may be given symbolically by writing

$$k_R = (\omega/c_o)\ f(k_I c_o/\omega,\ B,\ A), \tag{3.11}$$

since a unique value of k_R may be obtained for every value of k_I between 0 and ∞. As indicated in the above expression, the path of the line depends on B and A. When k_I is zero, k_R will be greater than ω/c_o. For sufficiently large B, k_R will initially decrease with increasing k_I and will reach a minimum which is between 0 and ω/c_o. In the asymptotic limit of large k_I, the path is given by the relation

$$k_R = (\omega/c_o)A^{-1}B^{-1}\log\,[2(k_I c_o/\omega)A^{-1}], \tag{3.12}$$

which describes a slowly increasing function. For smaller values of B, the line will dip only slightly to the left or not at all, and k_R will always be greater than ω/c_o.

Since the only admissible roots are those lying to the left of the branch line rising from ω/c_o, all of the roots of (2.10) in the

first quadrant will lie along the portion of the line for which $k_R < \omega/c_o$. This implies that there are no complex roots for small values of B, and that, for larger values, the roots all lie within a finite distance of the real axis--the maximum value of k_I being given approximately by

$$k_I = \frac{1}{2} A \, (\omega/c_o) e^{AB}. \tag{3.13}$$

We include a plot of the line of roots for several values of B and the particular case $A^2 = 0.25$ to illustrate these remarks (Fig. 4). We have plotted $k_R c_o/\omega$ vs. $k_I c_o/\omega$ as we wish to show the path of the line relative to the branch line.

As in the case of the real roots, each complex root may be associated with a mode number m. The m-th root will lie on the intersection of the line of roots with a second line,

$$k_R = (\omega/c_o)\, g_m(k_I c_o/\omega, A), \tag{3.14}$$

where g_m does not depend on B. This line intercepts the real axis at some point between ω/c_o and ω/c_1. For large values of k_I it has the asymptotic form

$$k_I = \frac{(\omega/c_o)(m\pi/2)(k_R c_o/\omega)}{\log[2A^{-1} k_R c_o/\omega]}, \tag{3.15}$$

which describes a line of slowly decreasing slope--k_I increasing nearly linearly with increasing k_R. Its behavior for intermediate values of k_I is difficult to describe by analytic methods. However, the path of the line may be found for any desired m and A by

a straightforward numerical computation. In Fig. 5 we show the results of one such calculation for several values of m and $A^2 = .25$. It is seen that, although the line starts and ends at the right of the branch line, it dips to the left, so that k_R is less than ω/c_o for a certain range of k_I.

The intersection of the lines (3.11) and (3.14) will be at most in only one point. They will intersect only if B is less than some value which depends on m. This value is approximately given by $(m+1)\pi/2$, which affords a good estimate of the high frequency cutoff for the complex modes.

The intersection of the two lines will not lie to the left of the branch line unless B is between two limiting values. These two values may be obtained in principle by setting k_R equal to ω/c_o in equations (3.11) and (3.13) and then eliminating k_I. Two such roots for B will always exist, provided the line (3.13) lies in part to the left of the branch line. These two values of B may be identified as the high frequency and low frequency cutoffs (in units of $\bar{\omega}$) for the m-th mode.

Another numerical scheme may be devised for obtaining the variation of $k_I H/2$ and $k_R H/2$ with respect to B for each mode. This scheme is also based on the use of parametric equations. In Figures 6 and 7 we show the results of such a computation for $A^2 = 0.25$ and several values of m. The low and high frequency cutoffs are as indicated.

Our study thus indicates the following:

1. For any given frequency there will be a finite number of roots.

2. The roots will lie along a line, such that the root with lowest mode number will be nearest the real axis.

3. With increasing frequency the roots with low mode numbers will successively disappear and roots with high mode number will successively appear.

4. For a root associated with any given m, there is a nonzero lower bound for the distance of this root from the real axis.

IV. THE ASYMPTOTIC SOLUTION

Let us now consider the nature of the formal solution in the event that r, the horizontal distance from the source, is large.

Our first task will be to show that the normal mode solution is the dominant portion in this limit. This is not immediately clear for our particular choice of source and receiver positions, and some consideration should be given before concentrating on the normal mode solution.

The Direct and Reflected Waves

For large r, the net distance traveled by either the direct wave or the wave reflected from the near edge of the center layer is nearly the same. One finds, in particular that

$$R_I - R_D \simeq 2r^{-1}(|z| - H/2)(|z_o| - H/2), \qquad (4.1)$$

which decreases with r as r^{-1}. It follows that the direct wave and the reflected wave will arrive at virtually the same time. The time lag δt of the reflected wave is given by dividing (4.1) by c_o. Furthermore, the two waves will have almost the same amplitudes, that of the reflected wave being slightly less than that of the incident wave. Since the two waves are 180° out of phase, they will very nearly cancel.

The sum of the direct and reflected waves is given approximately by the expression

$$\frac{1}{R_D}\left[f(\tau) - f(\tau-\delta t)\right] + \frac{c_o \delta t}{R_D^{\,2}} f(\tau) , \qquad (4.2)$$

where we have abbreviated $\tau = t - (R_D/c_o)$. The second term decreases with distance as r^{-3} and is generally negligible at large r. In the event the function $f(\tau)$ is differentiable, the first term is

$$\frac{\delta t}{R_D} f'(\tau),$$

which decreases as r^{-2}. In any event it is clear that, for any realistic source function, the portion of the wave carried by the direct and reflected waves will be negligible compared to any wave which decreases as r^{-1} or slower at sufficiently large r.

The Lateral Wave

The lateral wave, which is given by (2.14), may be analyzed by standard methods.[1-4] We have followed the method attributed to Gazaryan which is discussed in detail by Brekhovskikh.[20]

It is to be noted that, in general, $U(\beta_o)$ is finite at $k = \omega/c_o$ and hence that the integrand in (2.14) vanishes at the beginning of the path for the k integration. This property was achieved in the early stages of our formulation when we extracted the direct wave and the wave reflected from the near edge of the center layer. Since the Hankel function decreases as $\exp(-k_I r)$ with increasing k_I, the principal contribution to the integral will come in the earliest portion of the path for large r. If one then makes approximations consistent with this observation, he may obtain an expression which is proportional to r^{-2}.

This procedure is invalid in general, however, since the coefficient obtained will be infinite for certain discrete frequencies

which correspond to the cutoff frequencies of the normal modes. When one considers the integration for such a frequency at the outset, he discovers that the resulting expression is proportional to r^{-1}.

A general expression for the k integration which is sufficiently valid for all frequencies may be derived by first approximating $U(\beta_o) - U(-\beta_o)$ for points on the branch line near the point $k = \omega/c_o$ by an expression of the form

$$U(\beta_o) - U(-\beta_o) = \frac{\eta \sqrt{k_I}}{\zeta + k_I}$$

(where η and ζ are independent of k) and then integrating over the path using this approximation. When one does this and assumes r to be large, he may obtain the following expression for the lateral wave:

$$\psi_L \doteq \frac{1}{4r} \operatorname{Re} \int_o^\infty g(\omega) e^{i\omega(r/c_o - t)} (DB \sin 2B - \cos 2B) \frac{K(W^2)}{W^2} d\omega, \qquad (4.3)$$

where B is the frequency in units of $\bar{\omega}$, and D is defined by (2.5). The function $K(W^2)$ is that defined by Brekhovskikh as

$$K(W^2) \doteq \frac{4iW^2}{\sqrt{\pi}} \int_o^\infty \frac{S^2 e^{-S^2} dS}{S^2 + iW^2} . \qquad (4.4)$$

Its argument is dependent on r and ω and may be taken approximately as

$$W^2 = \frac{1}{4} r AH^{-1} B \sin^2 2B, \qquad (4.5)$$

where A is defined by (3.3). The derivation of (4.3) is lengthy but straightforward if one follows the steps outlined by Brekhovskikh.

As may be seen from the asymptotic formulas given by Brekhovskikh, the quantity $K(W^2)/W^2$ is of negligible magnitude when W^2 is large. We note from (4.3) that, for large r, this will be true unless sin 2B is very small. Thus the significant contributions in the integration over ω come from those frequencies near the cutoff frequencies.

The preceding remarks suggest an approximate method of evaluating (4.3). One first decomposes the integration to a sum of integrals, the first proceeding from $\omega = 0$ to $\bar{\omega}\pi/4$, the second from $\bar{\omega}\pi/4$ to $\bar{\omega}3\pi/4$, etc. In each of the terms one may then transform the variable of integration to W. Since the range of W will be large in any one integral and the principal contribution comes from relatively small values of W, the limits of integration in each term may be effectively taken to be 0 and ∞ for $n = 1$, and $-\infty$ and ∞ for $n > 1$. Also, since the factors in the integrand other than $K(W^2)/W^2$ and $e^{i\omega(r/c_0-t)}$ will vary slowly with W, they may be replaced by their values at $W = 0$. In this manner one may obtain the following expression for the lateral wave:

$$\psi_L = 2 \text{ Re} \left\{ \frac{\bar{\omega}}{6r} \left(\frac{H}{rA}\right)^{1/3} g(0) \, F_1(T_1) \right.$$

$$\left. + \sum_{n=2}^{\infty} (-1)^n \frac{\bar{\omega}}{4r} \left(\frac{H}{rA}\right)^{1/2} [(n-1)\pi/2]^{-1/2} g(\omega_c) e^{i\omega_c(r/c_0-t)} F(T_n) \right\} , \qquad (4.6)$$

where $\omega_c = (n-1)\pi\bar{\omega}$ and

$$F_1(T) = \int_0^\infty \frac{e^{ix^{2/3}T}}{x^{1/3}} \frac{K(x^2)}{x^2} dx,$$

$$F(T) = \int_{-\infty}^\infty e^{ixT} x^{-2} K(x^2) dx,$$

$$T_1 = (r/c_o - t)\, \bar{\omega} \left(\frac{H}{rA}\right)^{1/3},$$

$$T_n = (r/c_o - t)\, \bar{\omega} \left(\frac{H}{rA}\right)^{1/2} [(1/2)(n-1)\pi]^{-1/2}.$$

All the integrals given above are absolutely convergent and bounded for all values of T_1 and T. Furthermore, for any realistic source function, $f(t)$, the sum in (4.6) is absolutely convergent.

Our analysis thus shows that the lateral wave is composed of a term which decreases as $r^{-4/3}$ and a term which decreases as $r^{-3/2}$. It follows that the lateral wave will be negligible at large r when compared to a term which decreases as r^{-1}.

The Complex Mode Solution

The complex mode solution is composed of a sum of integrals over ω. The integral for the m-th mode when r is large may be written in the form

$$\frac{1}{\sqrt{r}} \int_{\omega_c}^{\omega_c'} P_m\, e^{i(k_m r - \omega t)} d\omega, \tag{4.7}$$

where P_m is a bounded function which does not depend on r. Since

the imaginary part of k increases with decreasing frequency, the major contribution to the integral comes from frequencies close to the high frequency cutoff ω_c'.

We noted in the last section that generally at the high frequency cutoff, the imaginary portion of k is not zero. This is true, for example, in the cases plotted in Fig. 7. The minimum value of k_I was nearly independent of mode number with a value of about 0.13(2/H). This indicates that the integral (4.7) must decrease exponentially with r and hence will be negligible for large r, when compared to any term decreasing as r^{-1} or slower. It follows that the same will be true for the complex mode solution as a whole.

The Normal Mode Solution

The portions of the wave discussed previously have all fallen off with distance faster than r^{-1}. As is well-known, the amplitude of the normal mode waves fall off as r^{-1} or slower (for the Airy phase). The normal mode solution is therefore the dominant portion of the solution.

Evaluation of the normal mode solution depends on carrying out the integrations over ω in (2.13). This integration may be performed to a good approximation by the method of stationary phase.[1-4,21] This method rests on the fact that the dominant contribution to the integral at large r comes from frequencies near those for which

$$\frac{d}{d\omega}(kr - \omega t) = 0 \qquad (4.8)$$

or

$$t = r/v_g, \tag{4.9}$$

where v_g is the group velocity defined as

$$v_g = (dk/d\omega)^{-1}. \tag{4.10}$$

If there are no frequencies satisfying (4.8), the integral will generally be of negligible value and may be approximately set to zero.

The group velocity depends on ω and the mode number n:

$$v_g = v_n(\omega). \tag{4.11}$$

In the next section we shall show that $v_n = c_o$ at the cutoff frequency, that it initially decreases to a minimum value less than c_1 and then increases monotonically, asymptotically approaching c_1 as $\omega \to \infty$. This is identical behavior to that found by Pekeris[1] for his model, and we shall follow his general method. We shall denote the minimum value of the group velocity for the n-th mode as $v_{n,o}$ and the corresponding values of ω and k by $\omega_{n,o}$ and $k_{n,o}$. Similarly, $k'''_{n,o}$ will represent the third derivative of $k_n(\omega)$ with respect to ω at $\omega = \omega_{n,o}$. In general we shall use primes to denote the derivatives of k_n with respect to ω.

The solutions of (4.8) shall be denoted $\omega_{n,g}(\tau)$ if $\omega < \omega_{n,o}$ and $\omega_{n,w}(\tau)$ if $\omega > \omega_{n,o}$, where $\tau = c_o t/r$. (The subscripts g and w stand for "ground" and "water," respectively, and were chosen in order to make use of the terminology introduced by Pekeris.)

Recalling our previous remarks about the behavior of the group velocity, we see that $\omega_{n,g}$ is defined for all τ between 1 and $\tau_{n,f} = c_o/v_{n,o}$, while $\omega_{n,w}$ is defined for all τ between c_o/c_1 and $\tau_{n,f}$.

By use of the method of stationary phase, a typical integral,

$$\varphi_n = \int_{\omega_c}^{\infty} i\, GH_o^{(1)}(k_n r) g(\omega)\, e^{-i\omega t}\, d\omega, \tag{4.12}$$

appearing in (2.13) becomes, for sufficiently large r:

$$\varphi_n = 0 \qquad \tau < 1; \tag{4.13a}$$

$$\varphi_n = \varphi_{n,g}, \qquad 1 < \tau < c_o/c_1; \tag{4.13b}$$

$$\varphi_n = \varphi_{n,g} + \varphi_{n,w}, \qquad \frac{c_o}{c_1} < \tau < \tau_{n,f} - \epsilon_n; \tag{4.13c}$$

$$\varphi_n = \varphi_{n,Ai}, \qquad \tau > \tau_{n,f} - \epsilon_n; \tag{4.13d}$$

where the quantities $\varphi_{n,g}$, $\varphi_{n,w}$ and $\varphi_{n,Ai}$ are generically called the "ground wave," the "water wave," and the "Airy phase," respectively. The choice of the parameter ϵ_n is somewhat arbitrary--the prime restriction is that $\varphi_{n,g} + \varphi_{n,w}$ be approximately equal to $\varphi_{n,Ai}$ when $\tau = \tau_{n,f} - \epsilon_n$ and r is sufficiently large. A satisfactory choice is

$$\epsilon_n = 10\, c_o\left(\tfrac{1}{2}\, |k'''_{n,o}|/r^2\right)^{1/3}. \tag{4.14}$$

The mathematical expressions for $\varphi_{n,g}$ and $\varphi_{n,w}$ are as follows:

$$\varphi_{n,g} = \frac{1}{r} g_{n,g}(\tau)P_{n,g}(\tau) \left\{e^{i(k_n r-\omega t)}\right\}_{\omega = \omega_{n,g}(\tau)}, \qquad (4.15)$$

where

$$g_{n,g}(\tau) = \left\{g(\omega)\right\}_{\omega = \omega_{n,g}(\tau)}, \qquad (4.16)$$

$$P_{n,g}(\tau) = \left\{2(k_n|k_n''|)^{-1/2}G(k_n)\right\}_{\omega = \omega_{n,g}(\tau)} \qquad (4.17)$$

and

$$\varphi_{n,w} = \frac{i}{r} g_{n,w}(\tau)P_{n,w}(\tau) \left\{e^{i(k_n r-\omega t)}\right\}_{\omega = \omega_{n,g}(\tau)}. \qquad (4.18)$$

The quantities $g_{n,w}$ and $P_{n,w}$ are defined analogously to (4.16) and (4.17). The quantity $\varphi_{n,Ai}$ is given by the expression

$$\varphi_{n,Ai} = \frac{e^{i\pi/4}}{r^{5/6}} g(\omega_{n,o})R(\omega_{n,o}) \left\{e^{i(k_n r-\omega t)}\right\}_{\omega = \omega_{n,o}} A_i(X), \qquad (4.19)$$

where

$$R(\omega_{n,o}) = \frac{1}{(|k_{n,o}'''|/2)^{1/3}} \left(\frac{8\pi}{k_{n,o}}\right)^{1/2} G(k_{n,o}) \qquad (4.20)$$

and Ai(X) is the Airy function defined as

$$Ai(X) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(u^3/3 + Xu)}du. \qquad (4.21)$$

Its argument, X, is given by

$$X = [t - (r/v_{n,o})] \, (|k'''_{n,o}|r/2)^{-1/3} \, . \tag{4.22}$$

The method we have indicated for evaluating the integral (4.12) is approximate and is generally regarded as inadequate for times near r/c_o and r/c_1. However, for our particular model, $\varphi_{n,g}$ vanishes at $t = r/c_o$ and $\varphi_{n,w}$ vanishes at $t_1 = r/c_1$. We accordingly feel that the solution (4.13) gives a qualitatively correct value of the integral for the times when its value is largest. It will not give a true value of the amplitude of the first peak in the ground wave, but this will be of negligible magnitude at large r, even if one uses a more refined method.

With this approximation the normal mode solution becomes

$$\psi_{NM} = \frac{4\pi}{H} \sum_{n=1}^{\infty} \text{Re } \varphi_n(r,t). \tag{4.23}$$

The interpretation of this solution will be our principal goal in the remainder of this paper.

V. DISPERSIVE CHARACTERISTICS OF THE NORMAL MODES

The dispersive characteristics of the n-th normal mode are those quantities and functions derivable from $k_n(\omega)$ which are needed for the evaluation and interpretation of the quantity $\varphi_n(r,t)$ as given by equation (4.3) in the preceding section. A partial list of such dispersive characteristics may be given as follows:

1. The phase velocity $v_p(\omega) = \omega/k$
2. The group velocity $v_n(\omega)$
3. The minimum value $v_{n,o}$ of the group velocity
4. The frequency $\omega_{n,o}$ at which the group velocity is a minimum
5. The second derivative $k_n''(\omega)$ of k with respect to ω
6. The third derivative $k'''_{n,o}$ at $\omega = \omega_{n,o}$
7. The functions $\omega_{n,g}(\tau)$ and $\omega_{n,w}(\tau)$
8. The quantity $\tau_{n,f}$ which represents the time of arrival of the Airy phase in units of r/c_o.

These dispersive characteristics are inherent properties of the model and are independent of the positions of the source and receiver. In this section we shall study these characteristics in some detail--with particular emphasis on their dependence on mode number.

The function $k_n(\omega)$ is given by the parametric equations (3.9) and (3.10). As μ increases from 0 to $\pi/2$ both k and ω increase monotonically. It follows that $k_n(\omega)$ is a monotonic increasing function of ω.

It is possible in principle to obtain $k_n(\omega)$ by first solving (3.9) for μ in terms of ω and then substituting this into (3.10). This is rigorously valid as there is a one to one relationship between μ and ω over the range of interest. We may therefore write $k_n(\omega)$ formally as

$$k_n(\omega) = (\omega/c_o)\, T(\mu_n(B)), \tag{5.1}$$

provided it is remembered that μ and k_n are defined only for $B > \frac{1}{2}(n-1)\pi$. Here we have abbreviated

$$T(\mu) = (1 + A^2 \sin^2\mu)^{1/2}.$$

The phase and group velocities are then given formally by the expressions,

$$v_p = c_o/T \tag{5.2}$$

and

$$v_n = c_o \,[T + B(dT/d\mu)(d\mu_n/dB)]^{-1}, \tag{5.3}$$

where μ is to be evaluated at $\mu = \mu_n(B)$. The derivative $d\mu_n/dB$ in (5.3) may be expressed in terms of μ since

$$\frac{d\mu}{dB} = \left(\frac{dB}{d\mu}\right)^{-1} = \frac{\cos\mu}{1 + B\sin\mu}, \tag{5.4}$$

by differentiation of (3.9). With the use of this expression, we may obtain the following expression for the group velocity:

$$v_n = \frac{c_o T(\mu)}{1 + A^2 \sin\mu \frac{B + \sin\mu}{1 + B\sin\mu}}. \tag{5.5}$$

The group velocity corresponding to any desired frequency above cutoff may be obtained by evaluating (5.5) at $\mu = \mu_n(B)$.

An alternative point of view is to regard (3.9) and (5.2) as parametric equations for the phase velocity, and (3.9) and (5.5) as parametric equations for the group velocity. The functions $v_p(\omega)$ and $v_n(\omega)$ may be tabulated by letting μ range from 0 to $\pi/2$ and computing $\omega = B\,\bar{\omega}$ from (3.9) and then v_p and v_n from (5.2) and (5.5) for each value of μ.

In a manner similar to that used for deriving parametric equations for $v_n(\omega)$, we may derive parametric equations for the derivatives (of any order) of k_n or v_n with respect to frequency. In each such case, we may express the desired dependent variable as a function of μ and B, which may be written explicitly in terms of elementary functions. Such an equation, in conjunction with the equation (3.9) for ω in terms of μ, gives a parametric representation of the variation of the dependent variable with ω.

One such expression which will be useful in our analysis is that giving the first derivative of the group velocity. We find

$$\frac{dv_n(\omega)}{d\omega} = \frac{v_n^2 A^2 (A^2+1)\tan^3\mu}{\bar{\omega} T^3 B_\mu^3 c_o}\left[(x-\beta)^2 - \beta^2 - \eta\right], \tag{5.6}$$

where

$$B_\mu = (1+B\sin\mu)/\cos\mu, \tag{5.7}$$

$$x = B \cos \mu = \frac{1}{2}(n-1)\pi + \mu, \tag{5.8}$$

$$\beta = \frac{1}{2}(1+A^2)^{-1} \cos \mu \sin^{-3} \mu \; (1+A^2 \sin^4 \mu), \tag{5.9}$$

$$\eta = 2(1+A^2)^{-1} \cos^2 \mu \; (1+A^2 \sin^2 \mu), \tag{5.10}$$

and v_n is given by (5.5). Expressions for k_n' and k_n'' may be found directly from (5.5) and (5.6), since

$$k_n' = v_n^{-1} \tag{5.11}$$

and

$$k_n'' = -(v_n)^{-2}(dv_n/d\omega). \tag{5.12}$$

Expressions for higher order derivatives become more complicated. We have carried this as far as the seventh order but will not reproduce the results here. The only higher order derivative needed is k_n''', whose value at $\omega_{n,o}$ determines the amplitude of the Airy phase. Our expression for this derivative is given in Appendix B.

The Phase Velocity

The phase velocity as given by (5.2) is a monotonic decreasing function of μ. Since μ increases with frequency, the phase velocity will also decrease with ω. At $\mu = 0$ the phase velocity is c_o, while it is c_1 at $\mu = \frac{1}{2}\pi$. It follows that v_p decreases monotonically from c_o to c_1 as the frequency increases from ω_c to ∞. For frequencies very close to the cutoff frequency one may derive a power series expansion by finding the derivatives of v_p

with respect to ω in terms of μ and then setting $\mu = 0$. In this manner one finds

$$v_p/c_o = 1 - \frac{1}{2} A^2(\delta B)^2 + \frac{1}{2} B_c A^2(\delta B)^2 + \ldots, \tag{5.13}$$

where $B_c = \frac{1}{2}(n-1)\pi$ and $\delta B = B-B_c$. In the other limit of very large frequencies one may obtain an asymptotic expression for v_p by expanding it in terms of B^{-1}. This is accomplished by expanding B^{-1} in terms of $(\pi/2)-\mu$ using (3.9), and then inverting this series and substituting the resulting expression into (5.2). Doing this, we find

$$v_p/c_1 = 1 + \frac{1}{2} A^2(A^2+1)^{-1}(\tfrac{1}{2}n\pi)^2 B^{-2} + \ldots \quad . \tag{5.14}$$

By using these two limiting expressions and because v_p decreases with frequency, one may obtain a fairly accurate sketch of the function $v_p(\omega)$.

<u>The Group Velocity</u>

Analogous techniques may be applied to the study of the group velocity $v_n(\omega)$. For frequencies near cutoff it behaves as

$$v_n/c_o = 1 - B_c A^2 \, \delta B + \frac{1}{2} A^2(2B_c^2 A^2 + 3B_c^2 - 3)(\delta B)^2, \tag{5.15a}$$

while for large frequencies it behaves as

$$v_n/c_1 = 1 + \frac{1}{2} A^2(1+A^2)^{-1}(\tfrac{1}{2}n\pi)^2 B^{-2}. \tag{5.15b}$$

It is clear from these expressions that $v_n = c_o$ at the cutoff frequency and that it initially decreases with increasing frequency. For large frequencies v_n is less than c_1 and increases with increasing frequency--approaching c_1 as $\omega \to \infty$. It follows that v_n must have a minimum value somewhere which is less than c_1.

That $v_n(\omega)$ has only one minimum can be verified by studying equation (5.6). It is readily seen that $dv_n/d\omega$ is positive or negative if the quantity $(x-\beta)^2 - \beta - \eta$ is positive or negative since all the other factors are positive. This quantity in turn will be positive or negative if

$$x - \beta - (\beta^2+\eta)^{1/2} \tag{5.16}$$

is positive or negative. The quantity x increases with μ with a slope of unity. The quantity $\beta + (\beta^2+\eta)^{1/2}$ is infinite at $\mu = 0$ and will generally decrease with increasing μ, its value being zero at $\mu = \pi/2$. For large A^2 it may have a positive slope over a small range of μ. The maximum positive slope it can have, however, is less than 1/2. (This may be verified by a careful examination of the expressions for β and η.) It follows that the quantity (5.16) is a monotonic increasing function of μ. It is minus infinity at $\mu = 0$ and $n\pi/2$ at $\mu = \pi/2$. It therefore must be zero in one and only one place at some nonzero value $\mu_{n,o}$. It follows that $dv_n/d\omega$ is less than or greater than zero depending on whether ω is less than or greater than $\omega_{n,o}$, where $\omega_{n,o}$ is that obtained from (3.9) with $\mu = \mu_{n,o}$.

In Fig. 8 we give a schematic sketch of $v_n(\omega)$ with the characteristic behavior predicted above. The actual detailed shape is dependent on A^2 and n. A good quantitative specification of the shape of the group velocity curve may be attained by focusing one's attention on a relatively small number of parameters which characterize the curve. These are (1) $\delta B_{n,1}$, the frequency relative to cutoff (in units of $\bar{\omega}$) at which the group velocity is equal to c_1, (2) $\delta B_{n,o}$, the frequency relative to cutoff at which the group velocity is a minimum, (3) $v_{n,o}$, the minimum value of the group velocity, and (4) $v''_{n,o}$, the curvature at the group velocity minimum. (The first three of these parameters are illustrated in Fig. 8.) These four parameters, with the two equations (5.13) and (5.14), give a fairly complete specification of the function $v_n(\omega)$. The alternative would be to tabulate $v_n(\omega)$ numerically. This, however, would be a cumbersome way of depicting the variation of the group velocity curves with mode number.

One finds, by equating (5.5) to c_1, that the parameter $\delta B_{n,1}$ corresponds to a value $\mu_{n,1}$ which satisfies the equation

$$(\tfrac{1}{2}(n-1)\pi + \mu)\tan\mu = [(1+A^2 \sin^2\mu)/(1+A^2)]^{1/2}. \qquad (5.17)$$

For large n, the solution of this equation is asymptotically

$$\delta = (1+A^2)^{-1/2}(\tfrac{1}{2}(n-1)\pi)^{-1}; \qquad (5.18)$$

the solution for smaller values of n must be found numerically. Since, for small μ,

$$\delta B \simeq \mu + B_c \mu^2/2,$$

we find

$$\delta B_{n,1} \simeq \frac{(1+A^2)^{1/2} + \frac{1}{2}}{(\frac{1}{2}(n-1)\pi)(1+A^2)} \tag{5.19}$$

for large n. A plot of $\delta B_{n,1}$ versus n is given in Fig. 9 for the particular case of $A^2 = 0.5625$ (or $c_1/c_o = 0.8$). The shape of the curve for small values of n was found by computing $v_n(\omega)$ and graphically determining $\delta B_{n,1}$.

The characteristics of the group velocity curve at its minimum may be studied analogously. The value of $\mu_{n,o}$ may be found by solving the equation

$$\frac{1}{2}(n-1)\pi = - \mu + \beta + (\beta^2 + \eta)^{1/2}, \tag{5.20}$$

where β and η are given by (5.9) and (5.10). For large n, one finds

$$\mu_{n,o} = \left[\frac{2}{(1+A^2)\pi(n-1)}\right]^{1/3}, \tag{5.21}$$

and, therefore,

$$\delta B_{n,o} = \frac{1}{2} (\frac{1}{2}(n-1)\pi)^{1/3} (1+A^2)^{-2/3}, \tag{5.22}$$

and

$$v_{n,o} = \frac{c_1^2}{c_o} (1 + \frac{3A^2}{2(A^2+1)^{2/3}(\frac{1}{2}(n-1)\pi)^{2/3}}). \tag{5.23}$$

An asymptotic expression for $v''_{n,o}$ may also be derived by first expressing v''_n in terms of μ and then setting $\mu = \mu_{n,o}$ and taking the limiting form for large n. We find

$$v''_{n,o} = \frac{3c_o}{\bar{\omega}^2} \frac{(1+A^2)^{2/3}A^2}{(\frac{1}{2}(n-1)\pi)^{4/3}} \tag{5.24}$$

A simple scheme may be used for finding the variation of the quantities $\delta B_{n,o}$, $v_{n,o}$ and $v''_{n,o}$ with n which avoids the computation of the complete curve for each value of n. One lets μ run over a sequence of values from $\pi/2$ to 0. For each μ, he computes n from (5.20). He then uses this n to compute B_c and B from (3.9) and finds the difference $\delta B_{n,o}$. Then he uses these values of B and μ to find v_n and v''_n. The quantities δB, v_n, and v''_n when plotted versus n give the behavior of the desired characteristics with mode number.

In Figures 10, 11 and 12 we show the results of such a calculation for $A^2 = 0.5625$. Figure 10 shows the minimum group velocity decreasing monotonically with increasing n, approaching c_1^2/c_o as n approaches infinity. The quantities $\delta B_{n,o}$ and $v''_{n,o}$ do not behave as systematically. We see that $\delta B_{n,o}$ has a minimum value at $n \simeq 6$ and that $v''_{n,o}$ has a maximum value at $n \simeq 4$.

The quantity $k'''_{n,o}$ may be determined from $v''_{n,o}$, since

$$k'''_{n,o} = - v''_{n,o}/v_{n,o}.$$

For large n this is proportional to $n^{-4/3}$, as may be seen from (5.23) and (5.24).

<u>Variation of k" With Frequency</u>

The analysis of the quantity k_n'' may be accomplished by methods similar to those used in studying the group velocity. Since it is related to the derivative of the group velocity by the relation (5.12), it is clear that it must have one of the two general forms depicted in Fig. 13--depending on whether $n = 1$ or $n > 1$.

If $n = 1$, k_n'' is zero at $\omega = 0$ and initially increases with increasing frequency, such that

$$k'' \simeq \bar{\omega}^{-1} c_o^{-1} B A^2 [3 - (5/6)(3A^2+16)B^2] \qquad (5.25)$$

for small B. It reaches a maximum value which is of the order of magnitude of

$$\frac{2\,\bar{\omega}^{-1} c_o^{-1} A^2 \sqrt{6}}{\sqrt{5}\,(3A^2+16)^{1/2}}$$

and then decreases, reaching zero at $\omega = \omega_{1,o}$. It continues to decrease to some minimum value and then increases again, its asymptotic form being

$$k_1'' \simeq - c_1 \bar{\omega}^{-1} c_o^{-2} (\tfrac{1}{2} A\pi)^2 / B^3. \qquad (5.26)$$

If $n > 1$ the quantity k_n'' has a nonzero positive value at the cutoff frequency and decreases linearly until it reaches a minimum value which is negative and then increases again, its asymptotic form being

$$k_1'' \simeq -c_1\bar{\omega}^{-1}c_o^{-2}\left(\tfrac{1}{2}An\pi\right)^2/B^3. \qquad (5.27)$$

We shall be interested primarily in the behaviour of k_n'' for frequencies less than $\omega_{n,o}$. For large n the behaviour over this range of frequencies is given approximately by the formula,

$$k_n'' = \frac{B_c A^2}{c_o\bar{\omega}[1+B_c\delta B]^{3/2}}\left[1 - (1+A^2)B_c^{-1/2}(2\delta B)^{3/2}\right], \qquad (5.28)$$

where B_c is $1/2(n-1)\pi$ and δB is $B-B_c$. We may see from this equation that k_n'' is large when δB is zero but decreases rapidly with increasing frequency. If δB is much greater than B_c^{-1}, k_n'' is decreasing as $\delta B^{-3/2}$. When δB is zero, k_n'' is approximately proportional to n and when δB is unity, k_n'' is proportional to $n^{-1/2}$. The consequences of this type of dependence on n will be discussed in the next section.

Variation of the Dispersion Characteristics With τ

The parameter τ represents time in units of r/c_o and is formally related to v_n by the equation

$$\tau = c_o/v_n. \qquad (5.29)$$

Let us note that, using (5.5), we may also express τ in terms of μ. Thus the variation of any characteristic with τ may be found in the manner that one would find its variation with frequency. One expresses both τ and the dependent variable in terms of μ and has two parametric equations which may be used for either numerical computations or as a basis for deriving approximate expressions.

We summarize here some of the properties of the two functions $\omega_{n,g}$ and $\omega_{n,w}$. Most of these properties may be readily derived from the preceding discussions in this section.

For τ between 1 and c_o/c_1, $\omega_{n,g}$ increases from the cutoff frequency to $\omega_{n,1}$. For τ close to 1 we find

$$\omega_{n,g} = \omega_c + \bar{\omega}\, B_c^{-1} A^{-2}(\omega-1) \tag{5.30}$$

for $n > 1$, and

$$\omega_{n,g} = \bar{\omega}\, (2/3)^{1/2}\, A^{-1/2}(\tau-1)^{1/2} \tag{5.30b}$$

for $n = 1$. The frequency thus varies slowly with τ in this limit for large n.

For τ near c_o/c_1, $\omega_{n,w}$ varies as

$$\omega_{n,w} = \frac{(c_o/c_1)\bar{\omega}\, A(\frac{1}{2}n\pi)}{\sqrt{2}(1+A^2)^{1/2}(\tau-c_o/c_1)^{1/2}} \,. \tag{5.31}$$

If τ is close to $\tau_{n,f}$, we have

$$\begin{matrix}\omega_{n,g}\\ \omega_{n,w}\end{matrix} = \omega_{n,o} \mp \frac{\sqrt{2}c_o^{\;1/2}(\tau_{n,f}-\tau)^{1/2}}{(v''_{n,1})^{1/2}\;\tau_{n,f}}, \tag{5.32}$$

where $\tau_{n,f}$ is $c_o/v_{n,o}$. One may study the variation of this dependence with n by using the asymptotic formulae (5.23) and (5.24).

The quantity $\tau_{n,f}$ is asymptotically $(c_o/c_1)^2$. For large n it differs from this by a quantity proportional to $n^{-2/3}$, as may be seen from (5.23).

We shall also need to know how the phase velocity of the ground wave varies with τ. When $\tau = 1$, the phase velocity is c_o. For $n = 1$, $(c_o - v_p)$ is proportional to $(\tau - 1)$ for τ close to 1; for larger n, it is proportional to $(\tau - 1)^2$. The following equation gives a reasonable approximation for τ between 1 and $\tau_{n,f}$ if n is large:

$$v_p = c_o \left[1 - \frac{1}{2} A^2 \frac{(\tau - 1)^2}{B_c^2(1 + A^2 - \tau)^2}\right] . \qquad (5.33)$$

(The derivation is discussed in Appendix C.) It is clear that v_p decreases relatively slowly with τ if n (or B_c) is large.

One may readily show from (5.33) that, when $\tau = \tau_{n,f}$, the phase velocity differs from c_o by a quantity proportional to $n^{-2/3}$ in the limit of large n. It follows that the phase velocity of the ground wave is nearly constant in the limit of large n.

VI. THE EXCITATION AMPLITUDES

The normal mode solution given in Section IV may be interpreted as follows: The waveform at time $t = r\tau/c_o$ consists of a sum of waves of slowly varying frequency. The frequencies present are $\omega_{n,g}(\tau)$ and $\omega_{n,w}(\tau)$ for $n = 1,2,3, \ldots$. In the previous section, we showed how one may determine just what frequencies correspond to a given τ. These frequencies were independent of the location of the source and of the time dependence of the source.

The next question we consider is that of the relative amplitudes of the various waves. It is clear from equations (4.15 - 4.21) that these amplitudes depend on a number of factors. One of these is $g(\omega)$, the frequency spectrum of the source. Here we shall consider the factors other than $g(\omega)$, i.e., those comprising the <u>excitation amplitude</u> for that particular frequency.

The excitation amplitude for a particular frequency is given by different expressions, depending on whether the frequency is associated with the ground or water waves or with the Airy phase. We define the following quantity:

$$E_n(\omega) = c_o^{-1}\, G_n(\omega)(k_n|k_n''|)^{1/2} \tag{6.1a}$$

if $\omega_c < \omega < \omega_{n,o} - \alpha_n$ or $\omega > \omega_{n,o} + \alpha_n$, or

$$E_n(\omega) = (0.536)r^{1/6}\, c_o^{-1}(2\pi/k_{n,o})^{1/2}(2/|k_{n,o}'''|)^{1/3}G_n(\omega_{n,o}) \tag{6.1b}$$

if $|\omega - \omega_{n,o}| < \alpha_n$,

where

$$\alpha_n = (20)^{1/2}(2r^2|k'''_{n,o}|^2)^{-1/6}, \qquad (6.1c)$$

as the relative excitation amplitude of the n-th normal mode. It is a dimensionless, real, and positive quantity which may be used to determine the relative strengths of excitation for various frequencies. In conjunction with some knowledge of the source spectrum, the knowledge of the relative excitation functions for each normal mode will allow us to make some qualitative predictions of the waveform. In particular, we may predict just which frequencies associated with which normal modes will be dominant at any given time, and we may predict the times at which the wave amplitudes will be greatest.

We shall not concern ourselves with the relative phases of the various normal modes. These will vary erratically with distance. It is more instructive to examine those features of the wave which vary slowly with distance at large distances. For qualitative estimates we have replaced the Airy function by its maximum value, .536, in (6.1b). This will suffice when comparing the relative magnitude of the Airy phases to other portions of the wave.

The Functions $G(\omega)$

All dependence on the source and receiver positions in the relative excitation functions is contained in the functions $G(k_n) = G_n(\omega)$, which we defined in (2.12). If we use the definitions of μ in equations (3.4), the quantity G_n may be written as

$$G_n(\omega) = \frac{B \sin \mu}{1 + B \sin \mu} \cos^2 \mu \, e^{-DB \sin \mu}. \tag{6.2}$$

Since B may also be expressed in terms of μ, the equations (3.9) and (6.2) constitute a pair of parametric equations, and we may numerically tabulate $G_n(\omega)$ versus ω in the manner described previously for the dispersion characteristics.

If one were to carry out such a calculation he would find that $G_n(\omega)$ has the schematic form shown in Fig. 14. It is zero at the cutoff frequency, rises to a single maximum $G_{n,o}$ at $\delta B = \Delta_{n,o}$ and then decreases, asymptotically approaching zero with increasing frequency.

Analysis of the parametric equations shows that G_n behaves as

$$G_n(\omega) = B^2 \qquad n = 1 \tag{6.3a}$$

$$= B_c \delta B \qquad n > 1 \tag{6.3b}$$

for frequencies near cutoff, and as

$$G_n(\omega) = (\tfrac{1}{2}n\pi)^2 \, B^{-2} \, e^{-(B - \frac{1}{2}n\pi)D} \tag{6.4}$$

for very large frequencies. The behaviour for intermediate frequencies may be studied by concentrating on the four quantities $G_{n,o}$, $\Delta_{n,o}$, $\Delta_{n,1}$, and $\Delta_{n,2}$, which give the maximum value of G_n, and the frequencies relative to cutoff in units of $\bar{\omega}$ at which G_n is maximum or 1/2 its maximum value. (See Fig. 14.) The difference $\Delta_{n,2} - \Delta_{n,1} = BW_n$ defines the <u>band width</u> carried by the n-th normal mode.

For large n, the quantity $G_{n,o}$ is independent of n and coincides with the maximum of $x(1+x)^{-1}e^{-Dx}$. For D = 1, this maximum is about 0.2. For larger D it decreases as D^{-1}, while for smaller D it increases, and asymptotically approaches 1 for very small D. In this limit we find that $\Delta_{n,o}$, $\Delta_{n,1}$, and $\Delta_{n,2}$ are inversely proportional to B_c(or n); the coefficients will depend on D. For D = 1, we find the following:

$$\Delta_{n,o} = 0.809\ B_c^{-1}, \tag{6.5a}$$

$$\Delta_{n,1} = 0.15\ B_c^{-1}, \tag{6.5b}$$

$$\Delta_{n,2} = 3.53\ B_c^{-1}, \tag{6.5c}$$

$$BW_n = 3.38\ B_c^{-1}. \tag{6.5d}$$

The coefficients will decrease roughly as D^{-1} for larger values of D.

Another quantity of interest is the value of $G_n(\omega)$ when $\omega = \omega_{n,o}$. If we substitute (5.21) and (5.22) into (6.2), we find for large n that

$$G_n(\omega_{n,o}) = \exp\,[-\ B_c^{2/3}(1+A^2)^{-1/3}D - (1/6)(1+A^2)^{-1}D], \tag{6.6}$$

which decreases rapidly with increasing n. In a similar manner, we may find $G_n(\omega_{n,1})$ where $\omega_{n,1}$ is the frequency at which the group velocity is c_1. We find, in the limit of large n, the following expression:

$$G_n(\omega_{n,1}) = \frac{(1+A^2)^{1/2}}{1 + (1+A^2)^{1/2}}\ \exp\,[-\ D(1+A^2)^{-1/2}]. \tag{6.7}$$

In this limit, $G_n(\omega_{n,1})$ is independent of n.

The Quantity E_n^o

Let us define

$$E_n^o\ (\omega) = \frac{c_o^{-1} G_n(\omega)}{[k_n |k_n''|]^{1/2}} \quad . \tag{6.8}$$

This is equal to the relative excitation function everywhere except in the vicinity of $\omega_{n,o}$.

This quantity may also be expressed in terms of μ and may be tabulated by letting μ run through a sequence of values and computing ω and $G_n(\omega)$ for each μ. The schematic form of this function is illustrated in Fig. 15. It is zero at the cutoff frequency. For larger frequencies it increases to a maximum and then decreases. This decrease is interrupted by a narrow singularity at $\omega_{n,o}$. For frequencies greater than $\omega_{n,o}$, E_n^o decreases rapidly, asymptotically approaching zero.

One may study the function E_n^o by methods similar to those used previously. Our analysis reveals the following:

1. For very large frequencies, the asymptotic form is

$$E_n^o = \frac{B_c}{AB}\, e^{\,-\,D(B-\frac{1}{2}n\pi)} \quad . \tag{6.9}$$

2. In the limit of large n, the magnitude of E_n^o at its first maximum decreases with n as B_c^{-1}. For D = 1, the coefficient is approximately 0.53 A^{-1}.

3. This maximum occurs at a value of δB which in the limit of large n is proportional to B_c^{-1}. For the case D = 1, we find the limiting form to be

$$\delta B = 2.10 \; B_c^{-1} \; . \tag{6.10}$$

4. The minimum value of E_n^o between its maximum and the singularity at $\omega_{n,o}$ is very small and decreases markedly with increasing mode number. In the limit of large n, we find the value of the function at this minimum to be

$$E_1^o = \frac{(2/3)^{1/2} B_c^{11/12}}{A(1+A^2)^{11/12}} \; e^D \exp\left[-B_c^{2/3}(1+A^2)^{-4/3} D\right]. \tag{6.11}$$

5. The minimum occurs when ω is very close to $\omega_{n,o}$. For very large n, we find the difference to be proportional to $B_c^{-4/3}$:

$$\omega_{n,o} - \omega = \frac{\bar{\omega}(1+A^2)^{1/6}}{2B_c^{4/3} D} \; . \tag{6.12}$$

Amplitude of the Airy Phase

The relative excitation function $E_n(\omega)$ for the Airy phase ($\omega = \omega_{n,o}$) is proportional to $r^{1/6}$ and increases slowly with r. It is clear that, for very large r, the amplitude of the Airy phase will be greater than that of the maximum amplitude of the ground wave. It is instructive to see just how large r must be in order that this be true.

We set $E_n(\omega_{n,o})$ equal to E_n evaluated at the frequency for which the ground wave amplitude is the largest. This gives, for D = 1, in the limit of large n, the following approximate value for r:

$$\frac{r}{H} = \frac{9}{64\pi^3} \; \frac{(1+A^2)^{8/3}}{A} \; e^{(1+A^2)^{-1}} \; B^{-19/3} \; e^{6B^{2/3}(1+A^2)^{-1/3}}. \tag{6.13}$$

To get an idea of the magnitude of r, we consider the case $n = 10$, $A^2 = 0.25$. This gives r/H of the order of 10^6. We can conceive of no practical case in which the ratio would ever be this large. If, for example, one were to try to apply this model to sound propagation in the atmosphere, H would have to be of the order of 20 km, and r would be of the order of 500 times the circumference of the earth.

This demonstrates that, unless one is interested in very low frequency sources, the dominant waves carried will be the ground waves. We shall henceforth concentrate our attention on these waves.

Time Dependence of Wave Amplitudes

Having eliminated the Airy phases and water waves from our consideration, we shall consider only frequencies between the cut-off frequency and $\omega_{n,o}$ when considering any particular mode. With this restriction, we may speak of a single-valued function $E_n(\tau)$ for each τ. This function is obtained in principle by evaluating $E_n(\omega)$ at $\omega_{n,g}(\tau)$.

As before, we may describe this function by two parametric equations giving E_n and τ in terms of μ. These equations may then be used for numerical computations or as a basis of analytic studies. To demonstrate that numerical computations of this function are feasible, we give in Fig. 16 a plot of $E_n(\tau)$ for several values of n, for $D = 1$ and $A^2 = 0.5625$, which was computed in this manner.

The qualitative form of $E_n(\tau)$ is clear from our previous discussion of $E_n(\omega)$ and $\omega_{n,g}(\tau)$. It is zero at $\tau = 1$, rises to a

maximum, and then decreases. The decrease will be interrupted by a narrow singularity near $\tau_{n,f}$. However, this latter feature shall be ignored, since we are neglecting the Airy phase.

It is instructive to examine $E_n(\tau)$ in the limit of large n. In this limit we may derive a single closed expression for E_n. We find

$$E_n(\tau) = \frac{(\tau-1)}{B_c[A^2-(\tau-1)]^{3/2}} \exp\left\{-\,D(\tau-1)/[A^2-(\tau-1)]\right\}. \qquad (6.14)$$

The derivation of this expression is given in Appendix C.

The most interesting feature of (6.14) is that it depends on n only through the factor B_c^{-1}. Thus the time of maximum amplitude for a high frequency source is independent of mode number. We find this to occur when

$$\tau - 1 = A^2[1 + \tfrac{1}{2}D] + 3/4 - \left\{(\tfrac{1}{2}A^2D+3/4)^2 + DA^4\right\}^{1/2}. \qquad (6.15)$$

If A^2 is small (shallow wave guide) or D is large, this is approximately

$$\tau - 1 = A^2(1+(2/3)A^2D)^{-1}. \qquad (6.16)$$

In general, τ decreases if D is increased.

We may estimate the duration of the wave by setting the exponent in (6.14) equal to some constant and solving for τ. If we take this constant to be 10, we find that the value of E_n is

$$\frac{10(10+D)^{3/2}}{AD^{3/2}B_c} e^{-10} , \tag{6.17}$$

and

$$(\tau-1) = \frac{10A^2}{10+D} . \tag{6.18}$$

We recall, from our previous discussion, that the maximum value of E_n was of the order of $(B_c A)^{-1}$ for $D = 1$. The quantity (6.17) is clearly much less than this for any appreciable value of D. Thus, (6.18) affords a reasonable estimate of the duration of the wave.

Discussion

The general method we have used for studying the characteristics of the normal modes is based on the fact that we may formally describe the variation of any characteristic with respect to frequency or τ by giving two parametric equations. In addition to the feasibility of the method for numerical calculations, it has a distinct advantage as a starting point for analytical studies. One may qualitatively describe the variation of the characteristic by examining the form of the parametric equations. Thus, for example, we proved that the group velocity had only one minimum. When we asked more quantitative questions, the study of the parametric equations suggested means of deriving approximate answers to these questions. Formerly, some of these could only be answered after a lengthy numerical calculation. As an example, we derived an expression for the amplitude of the Airy phase and showed that this was negligible compared to the maximum amplitude in the ground wave for higher order modes.

VII. QUALITATIVE DESCRIPTION OF THE WAVEFORMS

In this section, we shall apply the results of the preceding sections to the interpretation of the waveforms which would be observed at relatively large distances.

Comparison of the Contributions from the Individual Modes

The contribution to the waveform from a single normal mode is a wave of slowly varying frequency. As we showed in the previous section, the wave is generally composed only of a ground wave. In this event, we may associate a single frequency with each value of τ (time in units of c_o/r). This was a direct result of placing source and receiver outside the guide. The amplitude of the Airy phase may be appreciable for the lowest order modes, but this amplitude decreases rapidly if one increases D.

The principal effect of source and receiver being outside the guide was to restrict the band width carried by any given mode. This is clear from our studies of $G_n(\omega)$ and $E_n(\omega)$ in the previous section. The exponential factor $e^{-DB \sin \mu}$ in the relative excitation amplitude decreased rapidly for frequencies above the cutoff frequency for almost any appreciable value of D. The band width, in the limit of large mode number, was very small, and proportional to n^{-1}. In this limit there would be very little overlap of the bands carried by different modes. However, some overlap would occur for the lowest order modes if D were not large.

The contribution from a given mode will depend on the frequency spectrum, $g(\omega)$, of the source. However, if $g(\omega)$ does not vary significantly over the band of frequencies carried by the

mode, it is clear that one may take $g(\omega)$ as being constant over this frequency band. This will be true for higher order modes especially. Then one may assume $g(\omega)$ to be equal to its value at the cutoff frequency as a reasonable approximation. For this to be valid for lowest order modes, we should assume that $g(\omega)$ does not vary significantly over frequency ranges of the order of $\bar{\omega}$. This would be true for any realistic time dependence of the source, provided that the signal duration near the source were less than $1/\omega$. For example, if $f(t)$ is $\sin \alpha t$ between $t = 0$ and $t = 2\pi/\alpha$, and otherwise zero, this criterion will certainly be true if α is of the order of $10\,\bar{\omega}$.

The time duration of the contribution from a given mode, at a given distance r, will be roughly independent of mode number. This is clear from equation (6.14), which shows that, in the limit of large n, the relative excitation amplitude factors into B_c^{-1} and a quantity that depends only on τ. This will not be true for the low order modes, as the signal duration will then be limited by the parameter $\tau_{n,f}$, which increases with mode number.

It follows, from (6.14) and our previous remarks concerning $g(\omega)$, that the amplitudes of the various normal mode waves, at any given time, will vary with n as $B_c^{-1}g(\omega_c)$, for sufficiently large n.

Dominant Frequencies

If one were to plot the absolute values of the wave amplitudes for the various modes, corresponding to a given τ, versus mode number, he would find that the resulting curve has a maximum for

some value of n. This value of n would give the dominant mode corresponding to that particular τ. The frequency $\omega_n(\tau)$ would be the dominant frequency at that given value of τ. This frequency will clearly not be that frequency for which the absolute value of $g(\omega)$ is a maximum.

In the event that $g(\omega)$ has its maximum at a frequency large in comparison to $\bar{\omega}$, and varies relatively smoothly with frequency, our previous remarks indicate that this dominant frequency will be less than that for which $g(\omega)$ is maximum. As an example, we consider the hypothetical case

$$|g(\omega)| = \omega^2 e^{-(\omega/Q)^2}. \tag{7.1}$$

The maximum value occurs when $\omega = Q$. We assume $Q >> \bar{\omega}$. It is clear that any source with such a spectrum will primarily excite modes of high order. Thus, one may use (6.14) and approximately take the amplitude of the n-th normal mode as being proportional to $B_c^{-1}|g(\omega)|$ or $\omega_c e^{-(\omega_c/Q)^2}$, where ω_c is the cutoff frequency of the n-th normal mode. The dominant normal mode is then that whose cutoff frequency is approximately $Q/\sqrt{2}$. The dominant frequency will be of this order of magnitude and hence significantly less than Q, the dominant frequency in the source spectrum.

It follows that the signal observed at large distances will give the appearance of being composed of frequencies from the low frequency spectrum of the source. The higher frequencies will appear to have been "eliminated."

The above remarks afford us with a certain qualitative insight into the gross features of the waveforms. Naturally, no single mode will dominate, but, rather, a group of modes will be more predominant than others.

<u>The Empirical Phase Velocity</u>

The definition of phase velocity, as usually given, is applicable only to a single normal mode. However, one may give an empirical definition of phase velocity as the apparent velocity with which characteristic features in the waveform move in the horizontal direction. This definition is often applied in experiments on anomalous sound propagation in the atmosphere to determine the angle of incidence of the "ray" to the ground.[22,23] This definition will be meaningless, however, for long distance propagation, unless the distortion of the waveform with distance is small. This is clear since the waveform will have a long duration and it may be difficult to decide which peak in the waveform observed at one distance corresponds to a certain peak in the waveform observed at another distance. Unless there is a reasonable amount of correlation between waveforms observed at different distances, this question may be virtually impossible to answer.

Since each mode has a different variation of phase velocity with τ, one would expect, in the general case, when many modes contribute to the waveform, that the correlation between waveforms observed at different distances would be small, and that the empirical phase velocity would be ill defined. For the particular model considered in this paper, with source and receiver outside

the guide, the distortion of waveform with distance is relatively small. The reason for this circumstance is clear when one considers how the phase velocity varies with frequency. For any mode, the phase velocity is c_o at the cutoff frequency and decreases relatively slowly (as contrasted to the group velocity) with increasing frequency. Since, with source and receiver outside the guides, each mode carries only frequencies close to its cutoff frequency, the phase velocity will be very nearly constant for all frequencies of appreciable amplitude.

For each value of τ, there is an inherent uncertainty in the empirical phase velocity. If the source excites the modes for n between n_1 and n_2, the empirical phase velocity could range from $v_{n_2}(\tau)$ to $v_{n_1}(\tau)$, depending on just what characteristic feature in the waveform is being used as a basis for the measurement. The difference $v_{n_1} - v_{n_2}$ may be considered as a velocity of distortion. If observations are made at r and r+L, the correlation between these waveforms at corresponding times may be expected to be small if L is greater or of the order of magnitude of $v^2\omega^{-1}(v_{n_1} - v_{n_2})^{-1}$, where ω is the dominant frequency at the corresponding τ, and v is the average phase velocity of all the excited modes at the same τ. If L is this large, the phase difference between the n_1-th and n_2-th normal mode waves will have changed by roughly one radian.

We may therefore make a rough estimate of the distance the wave must travel before the waveform is significantly distorted. Assuming the source primarily excites high order modes, we use (5.33) to estimate the difference $v_{n_1} - v_{n_2}$. The critical factor in the

resulting expression will be $n_2 - n_1$. All other factors may be replaced by averages or order of magnitude equivalents without significantly changing L. Thus v_{n_1} and v_{n_2} are replaced by c_o, $n_1 + n_2$ is replaced by $4\omega/(\pi\bar{\omega})$, where ω is the dominant frequency of the source, etc. Furthermore, assuming that a large number of modes are excited, we may take $n_2 - n_1$ as being $(2/\pi)(\Delta\omega/\bar{\omega})$, where $\Delta\omega$ is the band width of the source spectrum. Doing this, we find

$$L \approx \frac{(1+A^2 - \tau)^2}{(\tau - 1)^2 A} \left(\frac{\omega}{\bar{\omega}}\right)^2 \left(\frac{\bar{\omega}}{\Delta\omega}\right) H \tag{7.2}$$

As would be expected, the length L is very large for τ close to 1 and decreases with increasing τ. It is large for higher frequency sources or sources of narrow band width.

<u>Variation of Waveform with Time at a Given Distance</u>

Since the frequency $\omega_n(\tau)$ for the ground wave increases with τ for each normal mode, it is clear that the earliest portions of the waveform must be characterized by lower frequencies than are present in the later portions of the waveform. It is interesting to note, however, that, for sources which primarily excite higher order modes, the frequency spectrum of the waveform is nearly uniform over the duration of the signal. This is clear from (6.14), which shows the variation of relative excitation amplitudes with τ to be independent of mode number.

Furthermore, for higher order modes, the frequency changes very slowly with τ. This is clear from (5.30). The waveform as a whole may be expected to have maximum amplitudes at the value of τ given by (6.16), since the relative amplitudes of all the higher order modes

will be maximum at this value. The variation of peak amplitudes with τ will be roughly as

$$\frac{(\tau-1)}{[A^2-(\tau-1)]^{3/2}} \exp\left\{-D(\tau-1)/[A^2-(\tau-1)]\right\}$$

throughout the duration of the signal. This also follows from (6.14).

If lower order modes are excited, their effect on the total waveform will be most noticeable during the earlier portion of the signal. This is clear from Fig. 16, which shows that the lower order modes have their peak amplitudes at earlier times than the limit represented by (6.16). At later times, these modes will not be present at all, since their duration is limited by $\tau_{n,f}$, which is significantly less than the maximum value of τ at which the signal may be observed.

VIII. SUMMARY AND CONCLUDING REMARKS

We have applied the method of normal modes to a simple model of layered media, for the particular case when both source and receiver were outside the layer with lowest sound velocity. The difficulties usually encountered in applying the method in guided wave problems were alleviated by the introduction of a variable μ. This enabled us to write the dispersion relations in a parametric form. Furthermore, it enabled us to answer some questions of theoretical interest, such as that of the location of the complex roots in the k-plane.

It was found that the variation of any characteristic of the normal modes with frequency could be described by two parametric equations in terms of μ. The parametric equations were found to be readily amenable to analytic investigation and we were able to study the variation of these characteristics with mode number. On the basis of this study, we were able to make some qualitative predictions of the nature of the waveforms which would be observed at large distances (Sec. VII) when a large number of modes were excited.

The methods we have used for studying the characteristics of the normal modes will, in general, be inapplicable to more complex models of layered media. It is hoped however, that the results obtained for this model may be of use in the development of a deeper insight into the nature of guided waves.

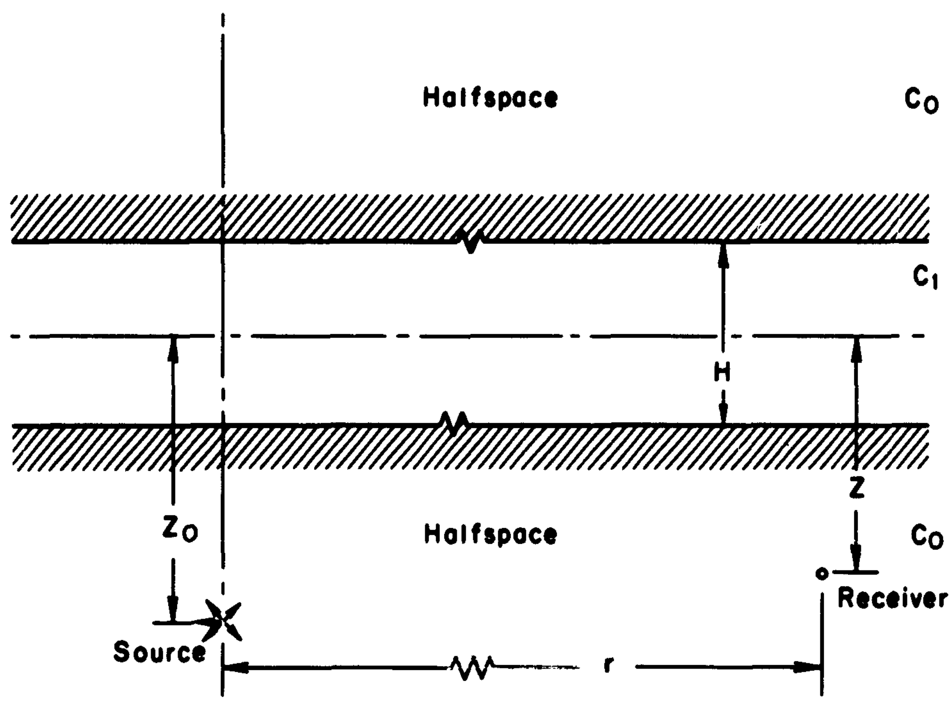

Fig. 1 - Diagram of the model considered in this paper.

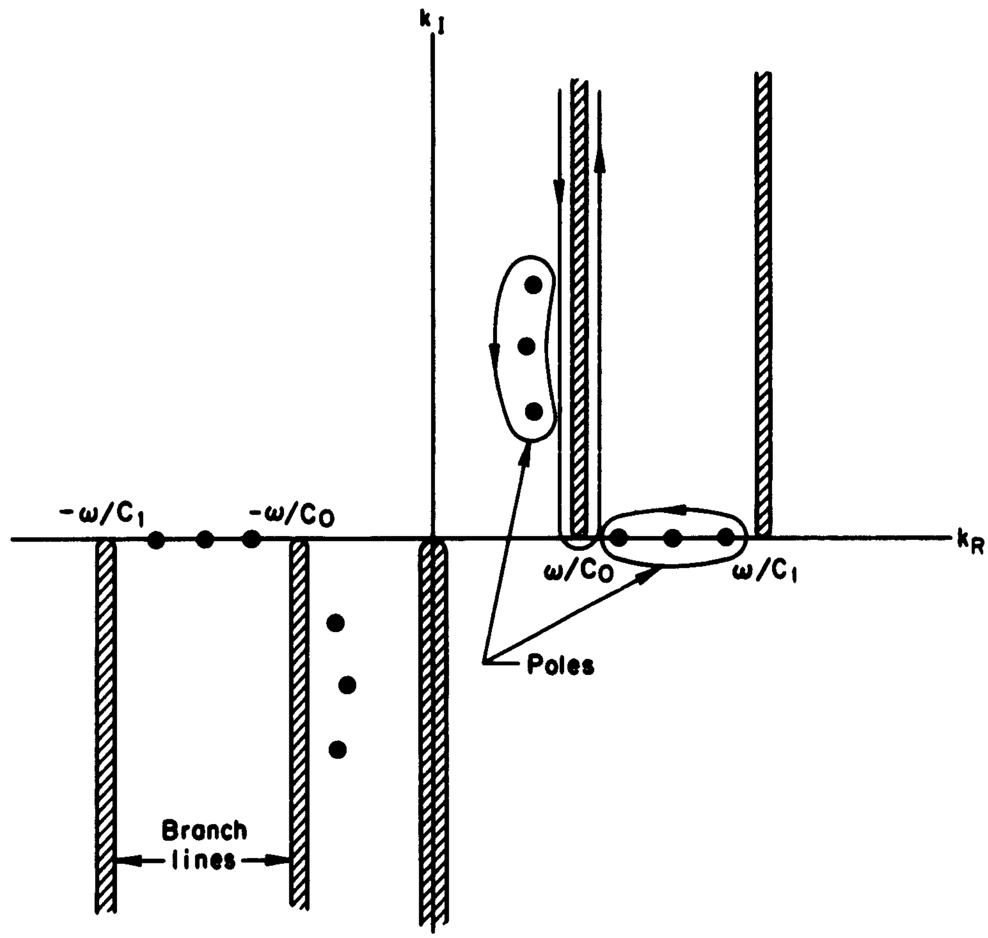

Fig. 2 - Diagram of complex k-plane showing deformed contour.

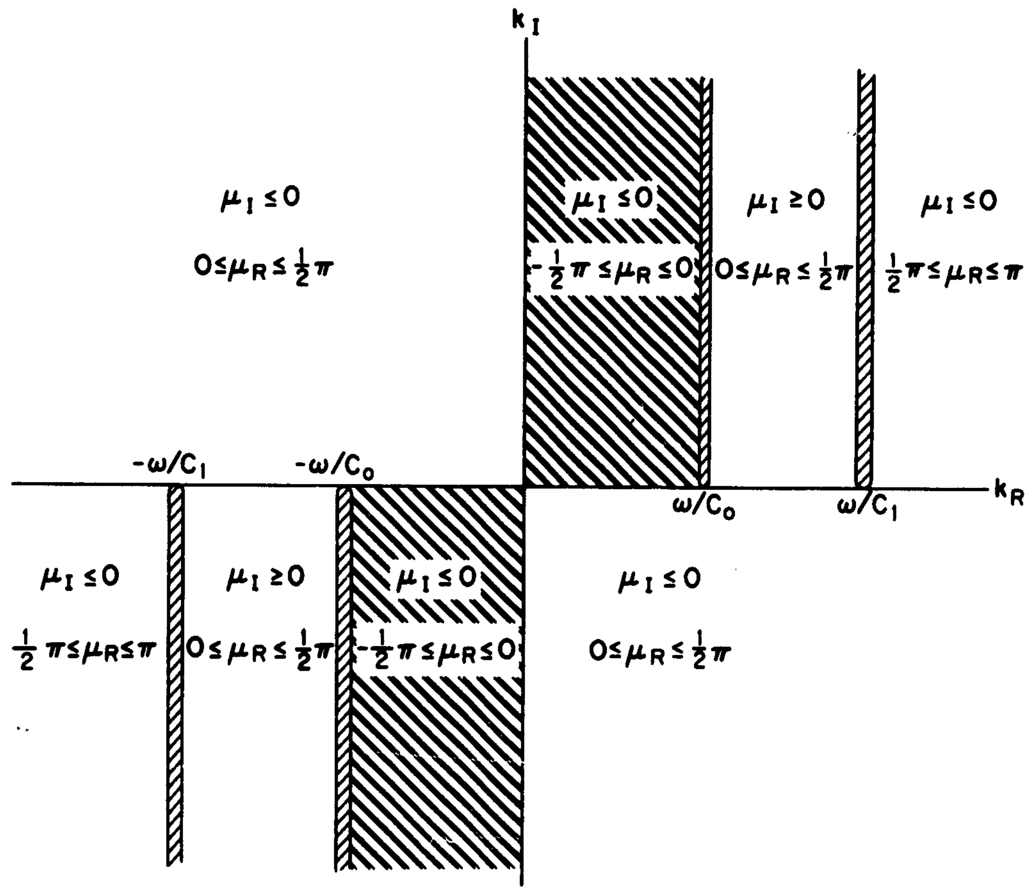

Fig. 3 - Allowed values of $\mu=\mu_R + i\mu_I$ in various regions of the k-plane. (It is assumed that $-\pi \le \mu_R \le \pi$.)

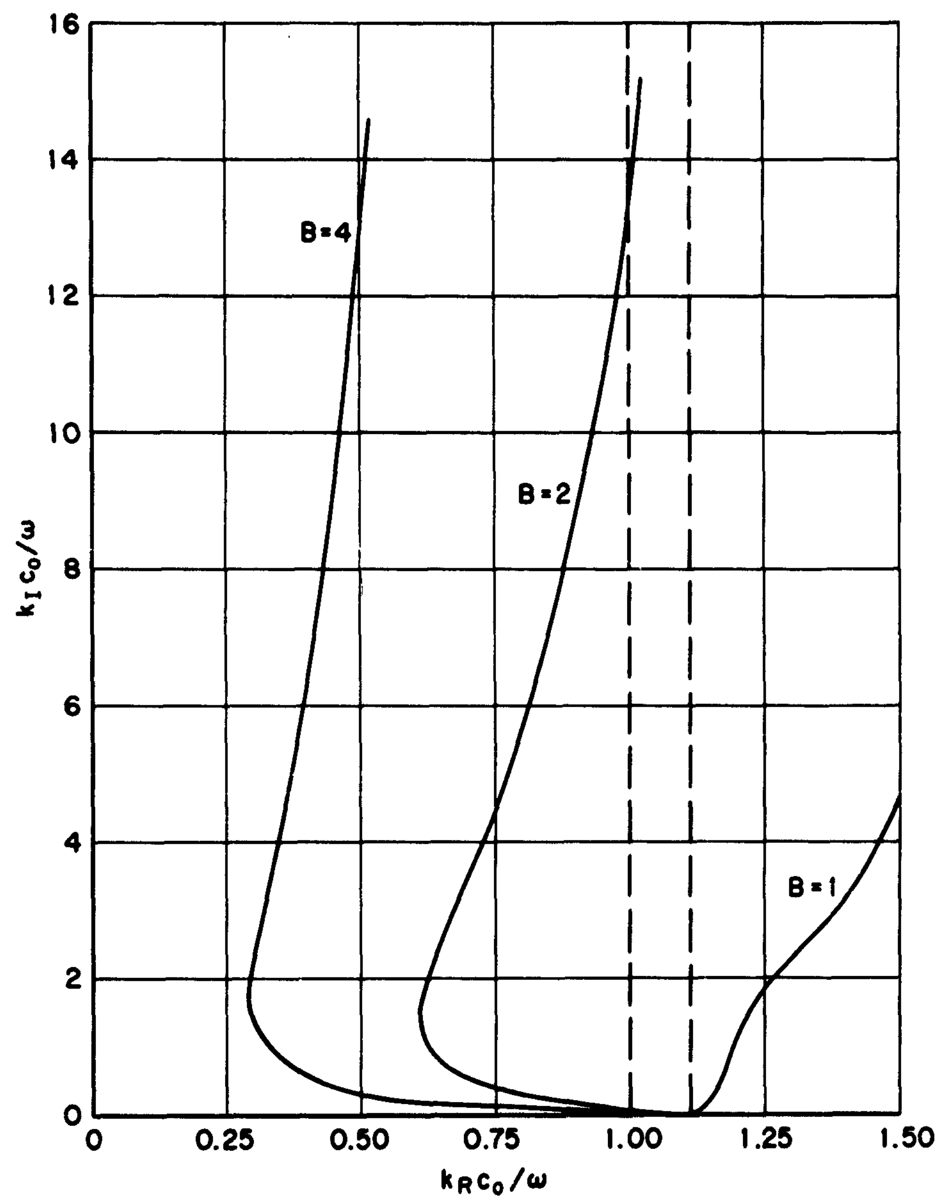

Fig. 4 - Plot of the line of roots in the k-plane for $A^2=0.25$ or $c_1/c_o=0.9$. (B is ω in units of $\bar{\omega}$.) The branch lines are indicated by the dashed lines.

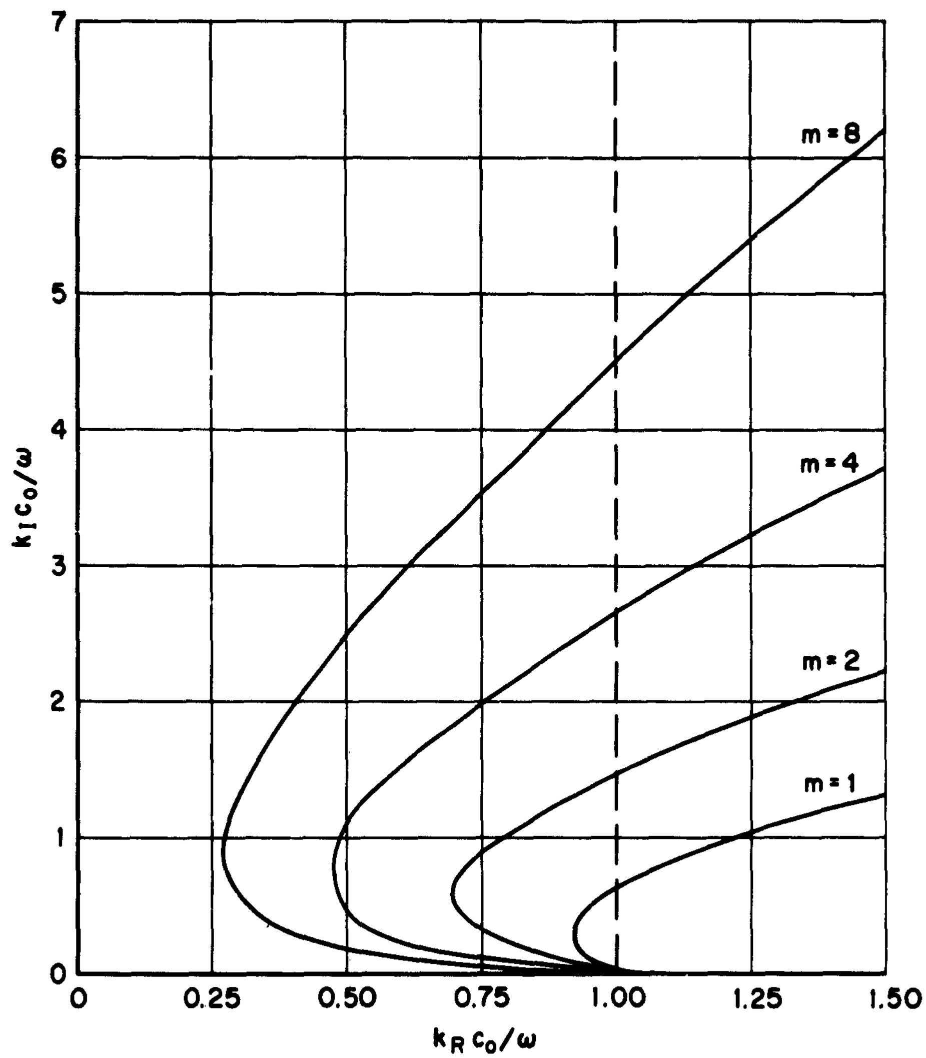

Fig. 5 - Plot of path traced by roots corresponding to given m in $k_R c_o/\omega$ - $k_I c_o/\omega$ plane as ω is varied. ($A^2 = 0.25$ or $c_1/c_o = 0.9$.)

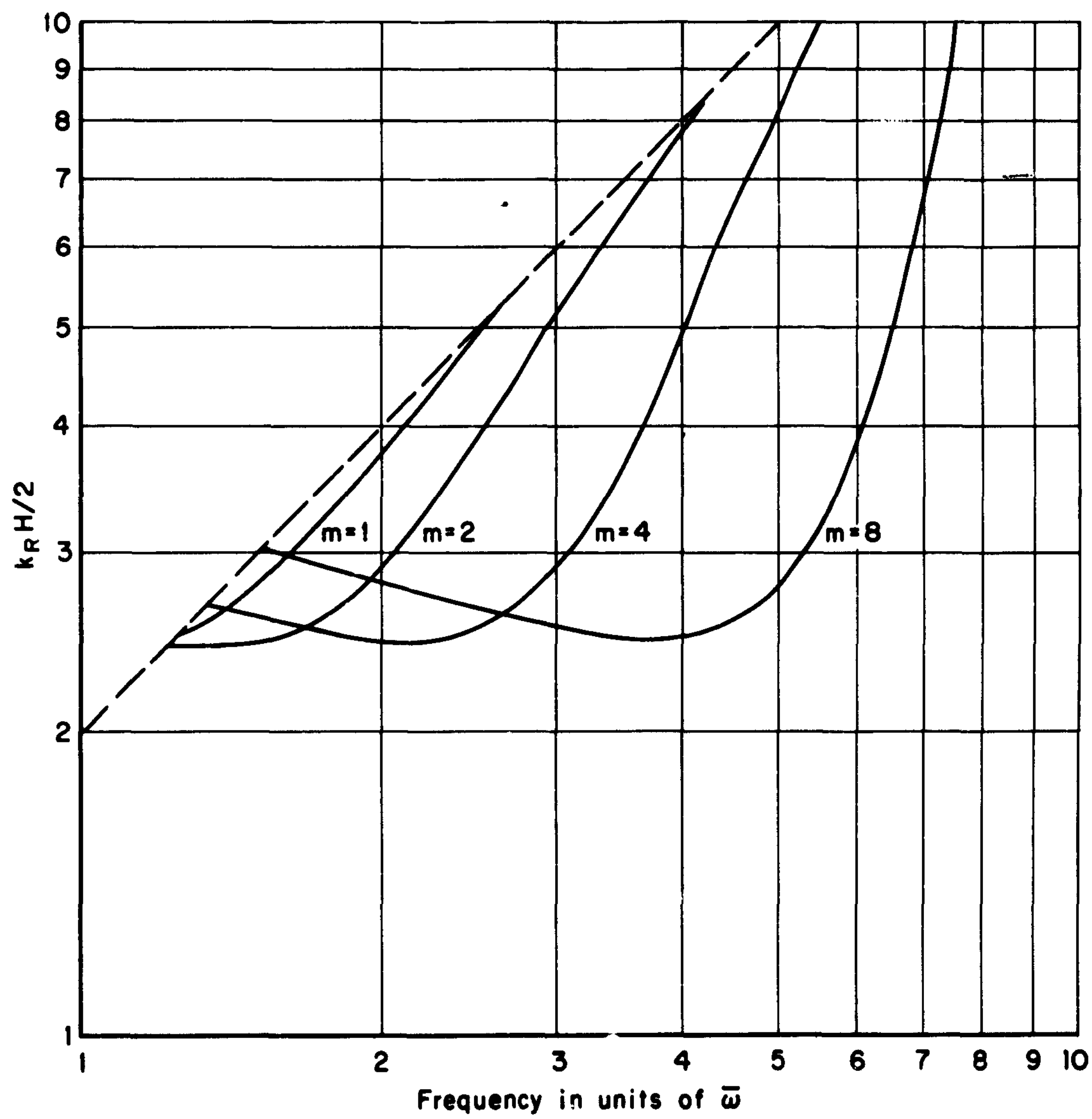

Fig. 6 - Plot of k_R(units of 2/H) vs. ω (units of $\bar{\omega}$) for the complex modes. (A^2 = 0.25.)

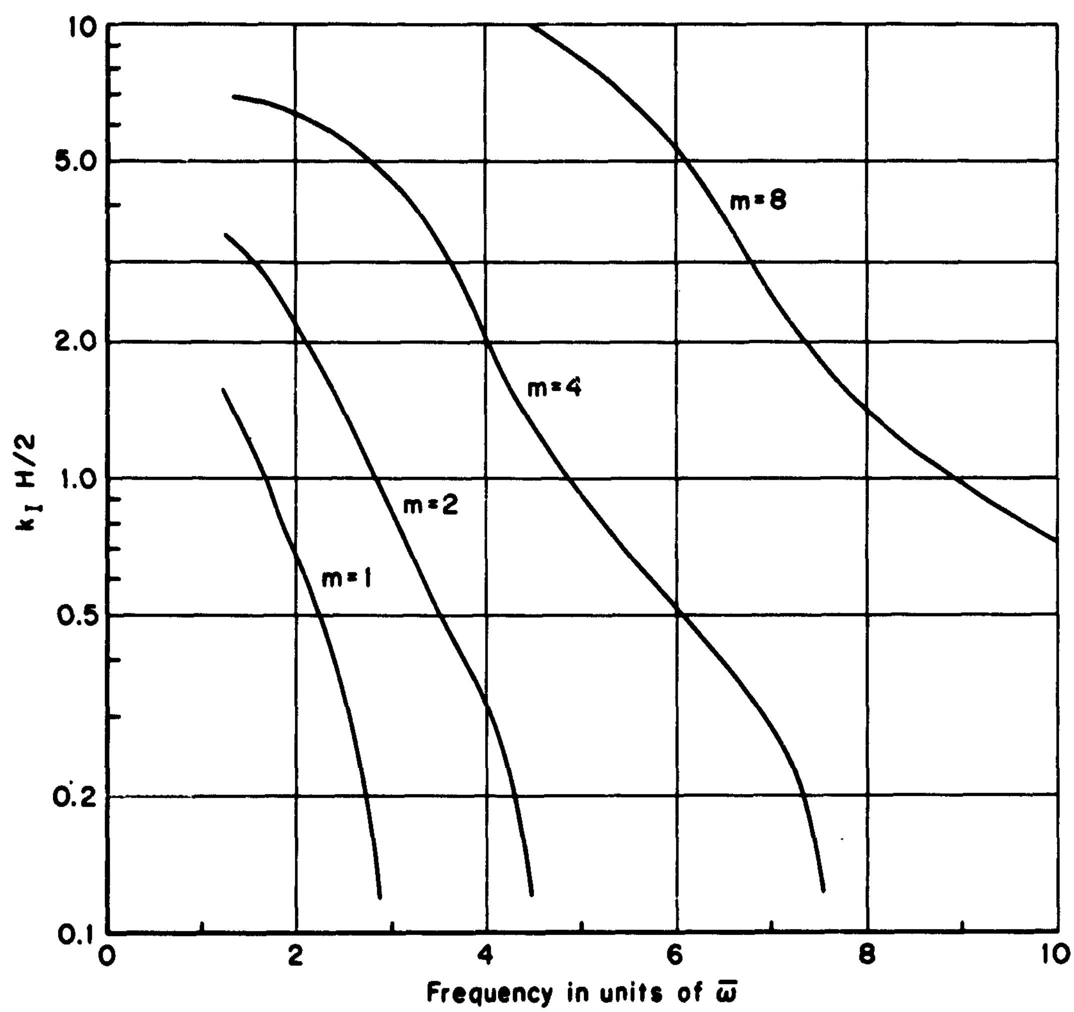

Fig. 7 - Plot of k_I (units of 2/H) vs. ω (units of $\bar{\omega}$) for the complex modes. (A^2 = 0.25.) The curves are shown only for frequenies between the two cut-off frequencies of the corresponding mode.

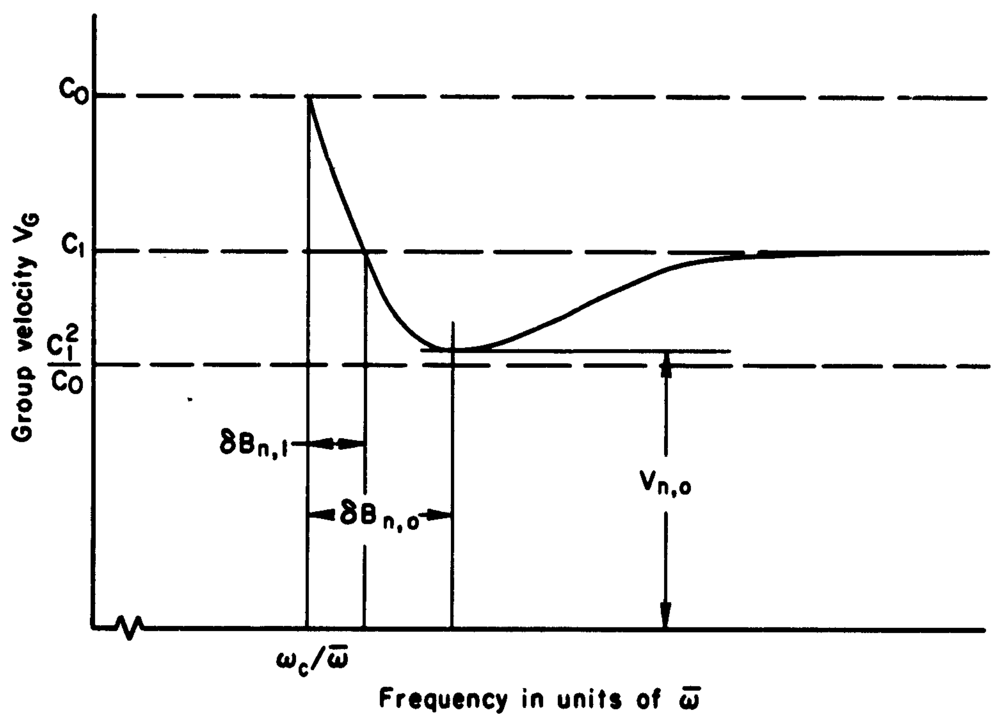

Fig. 8 - Sketch of group velocity vs. $B = \omega/\bar{\omega}$ showing characteristic behavior.

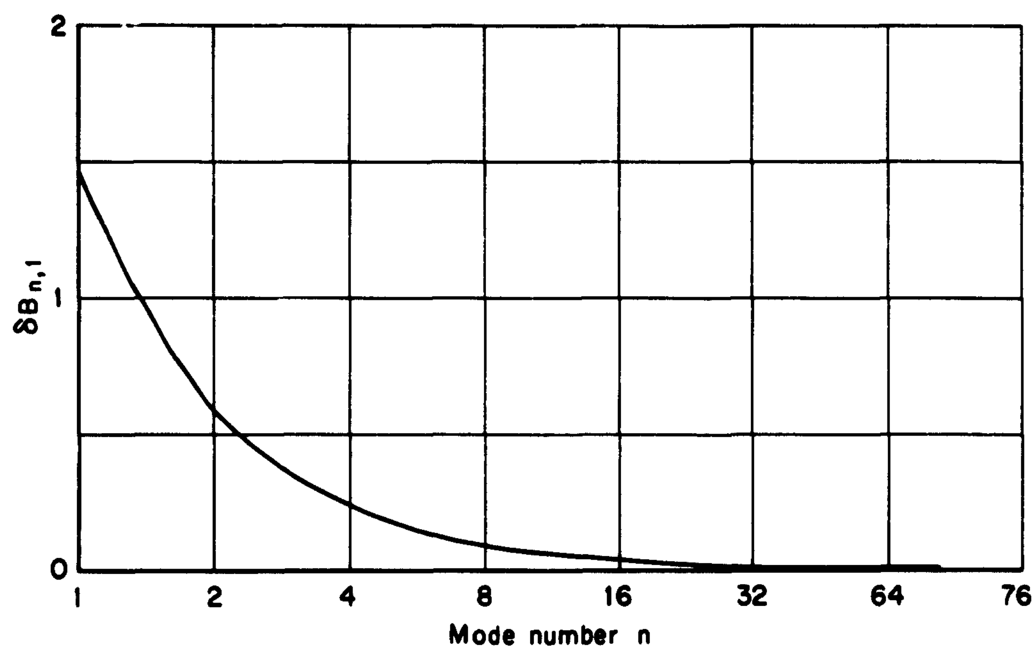

Fig. 9 - Plot of $\delta B_{n,1}$ vs. n for c_1/c_o = 0.8.

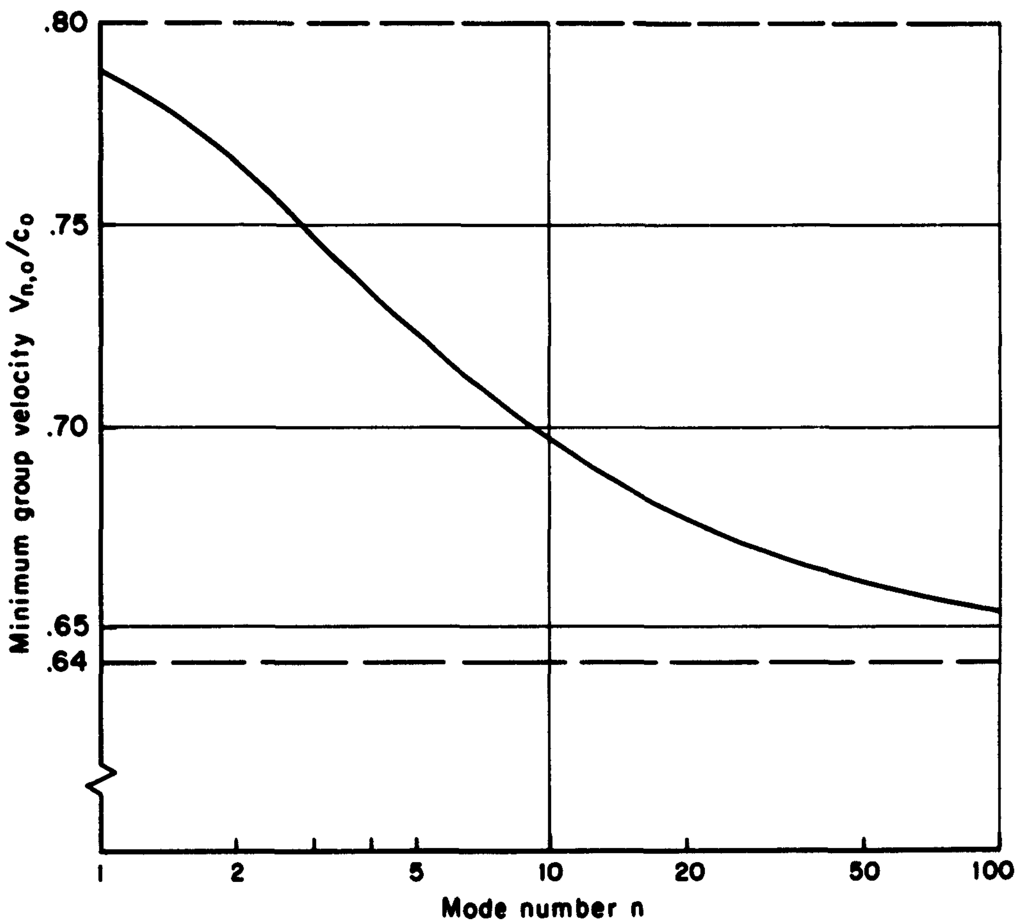

Fig. 10 - Plot of minimum value of group velocity (units of c_o) vs. mode number n. ($c_1/c_o = 0.8$.)

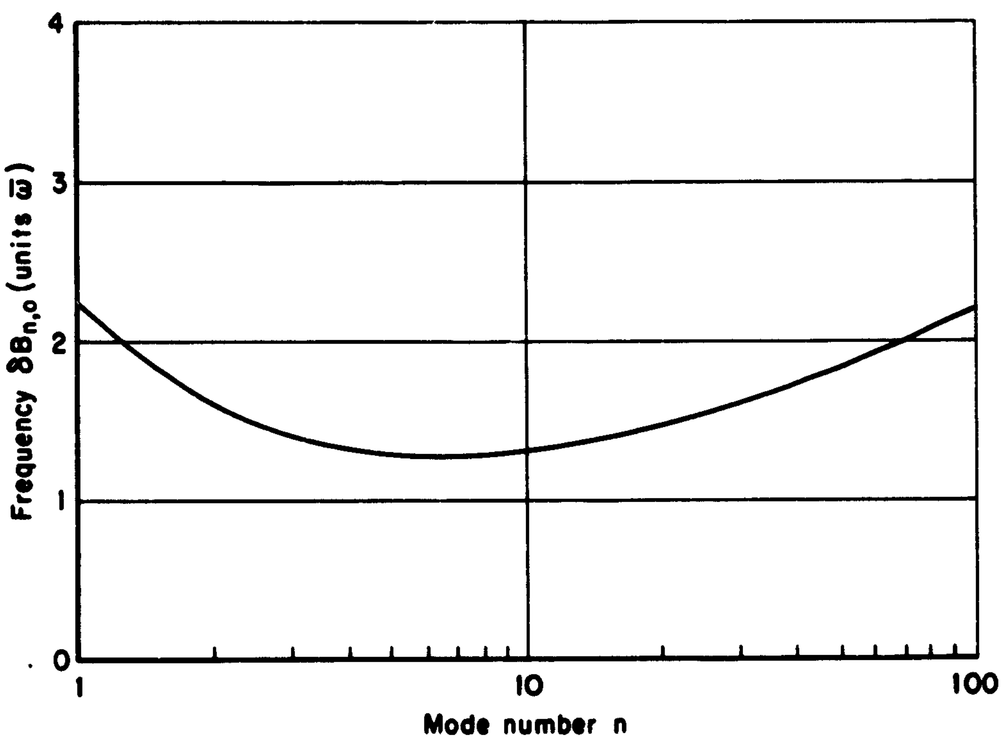

Fig. 11 - Frequency relative to cutoff frequency at group velocity minimum vs. mode number n. Frequency is in units of $\bar{\omega}$. (c_1/c_o = 0.8.)

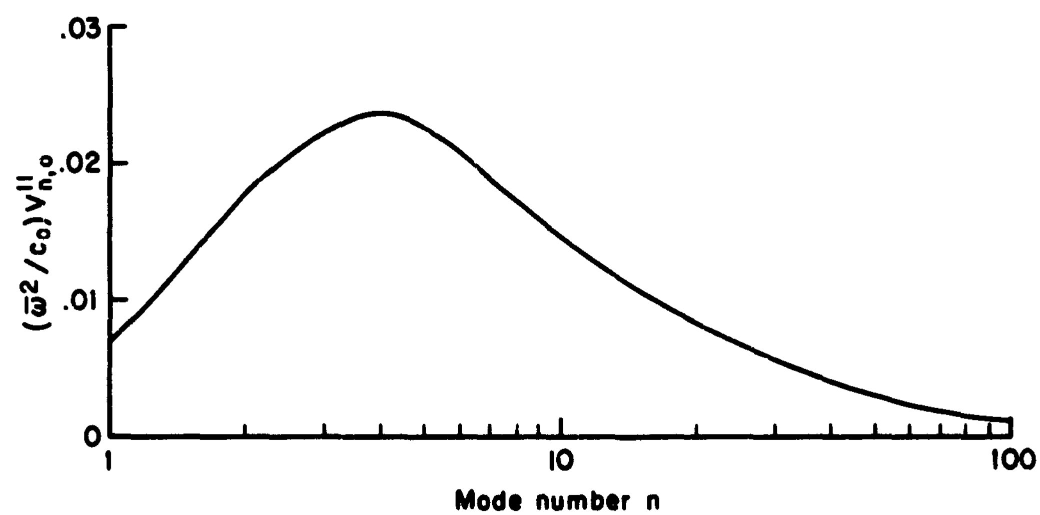

Fig. 12 - Plot of $v''_{n,o}$ (units of $c_o/\bar{\omega}^2$) vs. mode number n.
(c_1/c_o = 0.8.)

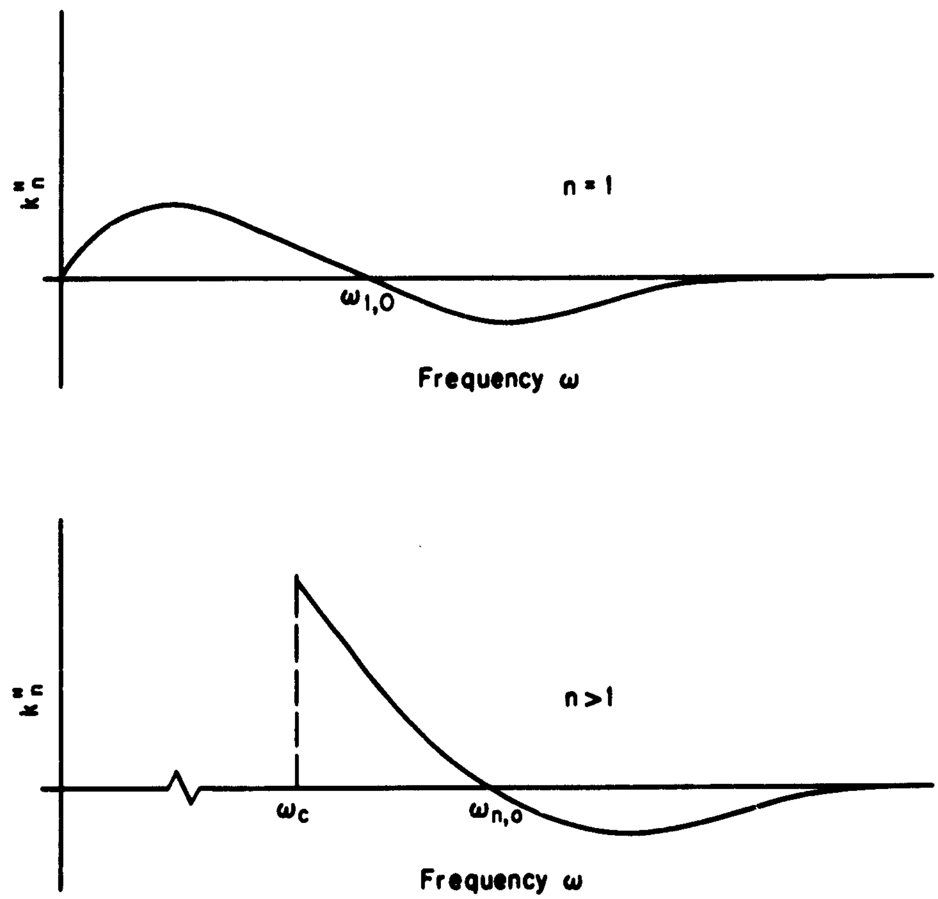

Fig. 13 - Sketch of the variation of $d^2k_n/d\omega^2$ with frequency for $n = 1$ and $n > 1$.

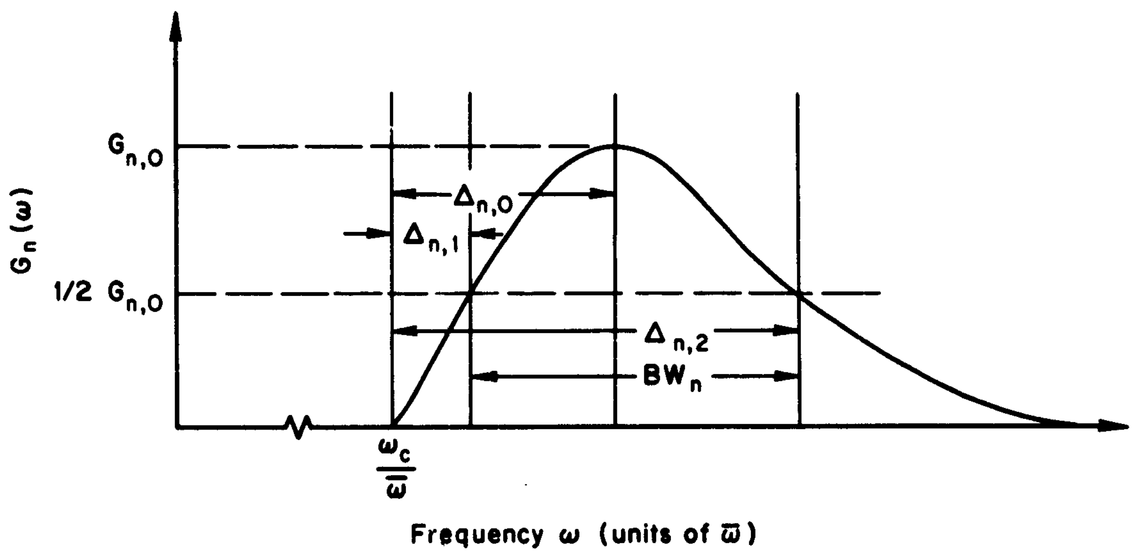

Fig. 14 - Sketch of the variation of $G_n(\omega)$ with $B = \omega/\bar{\omega}$.

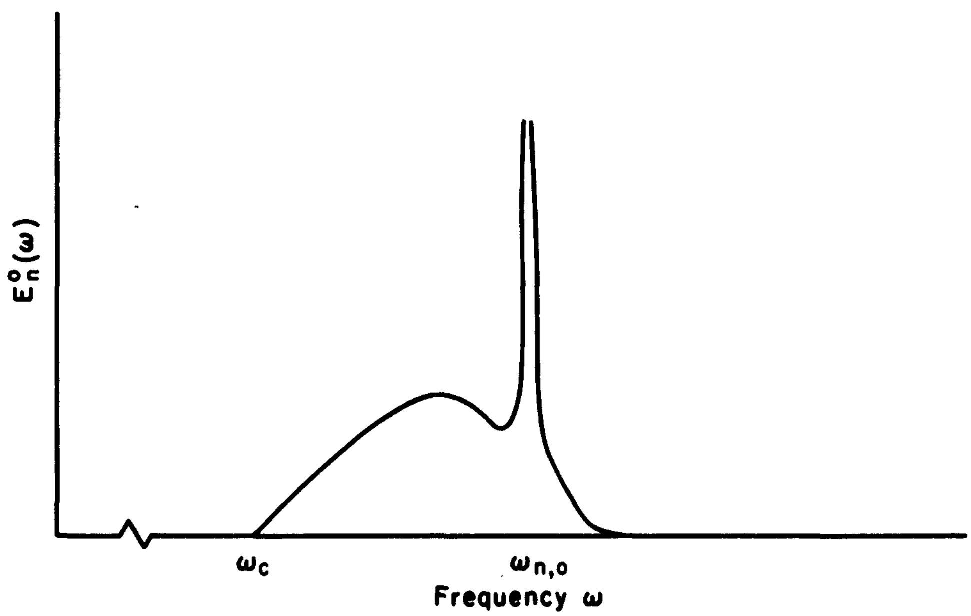

Fig. 15 - Sketch of the variation of $E_n^o(\omega)$ with frequency.

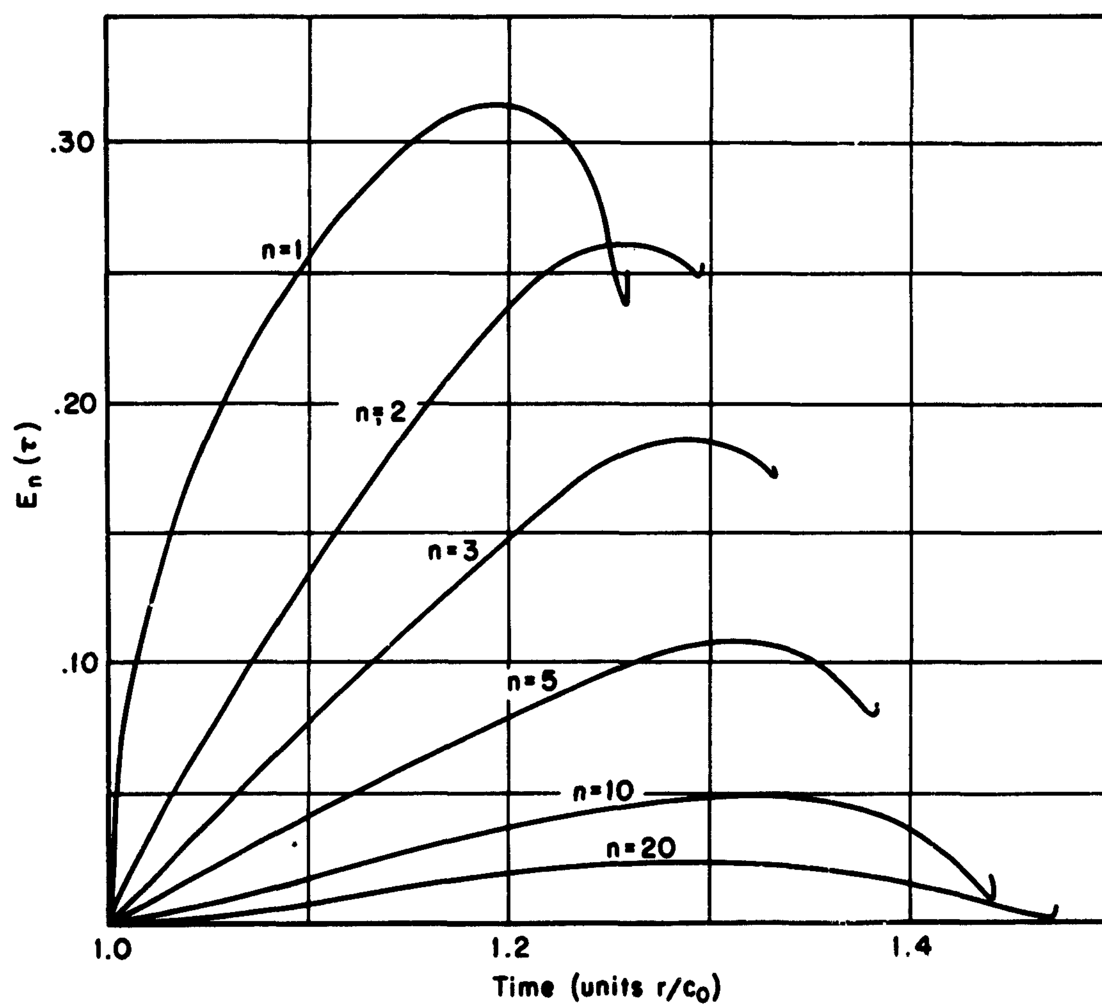

Fig. 16 - Plot of relative excitation function for various normal modes as a function of $\tau = tc_o/r$. ($D = 1.0$ and $c_1/c_o = 0.8$.)

APPENDIX A

Equation (3.5) is satisfied if and only if

$$B \cos \mu - \mu = \frac{1}{2}(m + 1)\pi \tag{A.1}$$

for some integer m. This is equivalent to two simultaneous equations if μ is complex. These two equations may be taken in the form:

$$\sin \xi = \frac{1}{B} \frac{\eta}{\sinh \eta}, \tag{A.2}$$

$$\tan \xi = \frac{1}{[\frac{1}{2}(m+1)\pi - \xi]} \frac{\eta}{\tanh \eta}, \tag{A.3}$$

where we have set $\mu = -\xi - i\eta$.

The first equation may be solved explicitly for ξ if the right hand side is less than 1. (If the right hand side is greater than 1, no solution will exist.) One may therefore use (A.2) and (3.6) to express k^2 in terms of η. Since k^2 is complex, we will have two parametric equations giving both $(k^2)_R$ and $(k^2)_I$ in terms of η. One may then let η run over a sequence of values and compute $(k^2)_R$ and $(k^2)_I$ for each η. Then one may compute the corresponding values of k_R and k_I from the relations

$$k_R, k_I = \left\{\pm \frac{1}{2}(k^2)_R + \frac{1}{2}[(k^2)_R{}^2 + (k^2)_I{}^2]^{1/2}\right\}^{1/2}, \tag{A.4}$$

where the plus sign is used to find k_R and the minus sign to find k_I. In this manner the line (3.11) may be tabulated.

The equation (A.3) may be easily solved numerically for ξ. This is done most simply by taking the arc tangent of both sides and then iterating, starting with any convenient value of ξ between 0 and $\pi/2$. The convergence will be rapid for $m \geq 1$. (One may show that there will be no solution of (A.3) for which $0 < \xi < \pi/2$ unless $m \geq 1$. The iteration method is therefore satisfactory for all cases of interest.) One may let η run through a sequence of values and compute ξ for each η. This value of ξ is then used to compute $B = \omega/\bar{\omega}$ from (A.2). Then $(k^2)_R$ and $(k^2)_I$ may be computed by use of (3.6). Finally, k_R and k_I may be computed by use of (A.4). For each value of η, one tabulates ω, k_R, and k_I.

The curves of k_R vs. k_I, or k_R vs. ω, or k_I vs. ω may then be plotted. This was the method used to find the curves shown in Figures 5, 6, and 7.

The two equations, (3.12) and (3.15), were obtained after an examination of the equations, (A.2), (A.3), (3.6), and (A.4), in the limit of large η.

APPENDIX B

The quantity k_n''' is given by the expression

$$k_n''' = c_o^{-1} \bar{\omega}^{-2} [3 T_{BB} + B T_{BBB}], \tag{B.1}$$

where

$$T_{BB} = \mu_B^2 T_{\mu\mu} + \mu_{BB} T_\mu, \tag{B.2}$$

$$T_{BBB} = \mu_B^3 T_{\mu\mu\mu} + \mu_{BBB} T_\mu + 3\mu_B \mu_{BB} T_{\mu\mu}, \tag{B.3}$$

$$T_\mu = A^2 (\sin \mu \cos \mu)/T, \tag{B.4}$$

$$T_{\mu\mu} = [A^2(1 - 2 \sin^2\mu) - T_\mu]/T, \tag{B.5}$$

$$T_{\mu\mu\mu} = - T_\mu (4T + 3T_{\mu\mu})/T, \tag{B.6}$$

$$T = (1 + A^2 \sin^2\mu)^{1/2}, \tag{B.7}$$

$$\mu_B = B_\mu^{-1}, \tag{B.8}$$

$$\mu_{BB} = - \mu_B^3 B_{\mu\mu}, \tag{B.9}$$

$$\mu_{BBB} = - \mu_B^4 B_{\mu\mu\mu} - 3\mu_B^2 \mu_{BB} B_{\mu\mu}, \tag{B.10}$$

$$B_\mu = (1 + B \sin \mu)/\cos \mu, \tag{B.11}$$

$$B_{\mu\mu} = B + 2B_\mu \tan \mu, \tag{B.12}$$

$$B_{\mu\mu\mu} = 3B_{\mu} + (3B_{\mu\mu} - B) \tan \mu. \tag{B.13}$$

To tabulate k_n''' vs. ω, one lets μ run through a sequence of values. For each value of μ, he computes B from (3.9) and then proceeds to calculate k_n''' by computing the relevant quantitites in the following sequence: T, T_{μ}, $T_{\mu\mu}$, $T_{\mu\mu\mu}$, B_{μ}, $B_{\mu\mu}$, $B_{\mu\mu\mu}$, μ_B, μ_{BB}, μ_{BBB}, T_{BB}, T_{BBB}, k_n'''. This scheme could probably be simplified, but doing this would require extensive algebraic manipulations. The method we outline above leaves the algebra to an electronic computer.

APPENDIX C

If $B_c\mu^3 \ll 1$, one may approximately take

$$\frac{c_o}{v_n} = \tau = 1 + A^2 \frac{B_c\mu}{1 + B_c\mu} \tag{C.1}$$

or, equivalently,

$$\mu = \frac{B_c^{-1}(\tau-1)}{A^2-(\tau-1)} . \tag{C.2}$$

If one examines the expression (5.2) for the phase velocity in the limit of small μ, he may obtain equation (5.33) using (C.2).

To derive (6.14), one first expresses $E_n(\omega)$ in terms of μ, assuming $\mu \ll B_c^{-1/3}$. One finds, in lowest order, that

$$E_n \approx (B_cA)^{-1} (1+B_c\mu)^{1/2} B_c\mu \, e^{-DB_c\mu}. \tag{C.3}$$

Equation (6.14) follows with the substitution (C.2).

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